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PREPARATIONS FOR THE NUMERICAL EVALUATION  
OF SECOND ORDER MOLODENSKY-TYPE FORMULAS

by

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The Ohio State University  
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## ABSTRACT

A set of formulas proposed by Moritz and constituting a second order solution of Molodensky's problem has been transformed in a way that the resulting formulas are numerically tractable. The difficulty in the original formulas arose from the occurrence of singular integrals. Their regularization introduces derivatives of gravity anomalies and terrain heights and necessitates a numerical differentiation procedure. Spline function interpolation has been chosen to deal with this. The existing truncation theory has been extended to cover more sophisticated truncation procedures as well as a succession of heavier and heavier smoothed versions of the integrand.

## FOREWORD

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## INTRODUCTION

Moritz (1969) proposed a set of formulas which, under certain assumptions, constitute a second order solution for Molodensky's problem.

These formulas are numerically not tractable because singular integrals and even iterated singular integrals occur. The occurrence of the singular integrals implies that the formulas are integro-differential formulas in disguise. The singularities can readily be removed by allowing derivatives to show up openly. Green's second formula for surfaces is the essential tool to regularize the integrals, and regularization should take place only within a small cap centered at the point of interest. For the outer zones the original form of the integrals is more advantageous since, in many cases the kernel tapers off quickly with increasing distance.

For remote areas a smoothed version of the involved functions like gravity anomalies and terrain heights can be used. The existing truncation theory has been reviewed and somewhat extended to cover more sophisticated truncation procedures as well as a succession of heavier and heavier smoothed versions of the involved functions. Appendix B, which is concerned with truncation, is expected to be of some interest beyond the immediate applications in this report.

The occurrence of derivatives in the modified formulas introduces the necessity of numerical differentiation. Assuming that the functions are only given at discrete locations or in block average form the spline interpolation approach has been chosen. This interpolation method has some advantages over the classical methods of polynomial or trigonometric interpolation. The bi-cubic spline interpolation procedure assumes that the functions are given at discrete locations forming a rectangular grid. These can be accomplished by prediction methods applied to an originally irregular pattern of sample points. In the case of block averages the most points of the blocks can be used to form the rectangular grid.

This work has been carried out under some time pressure. Therefore extensive numerical tests could not be performed. Some tests have been made with bi-cubic spline interpolation and also with one of the singular integrals.

The proposed methods are certainly time consuming. However a second order approach will be employed if in a limited area a good accuracy is needed. In this case the chosen approach appears to be computationally feasible.

# 1. THE ORIGINAL FORMULAS

We shall deal with equ. (8.1) to (8.3) in Moritz (1969). In somewhat different notation these formulas read as follows:

$$\begin{aligned} a(\xi) = & \frac{R}{4\pi G} \int_{\Gamma} \text{St}(\xi \cdot \eta) (\Delta g(\eta) + G_1(\eta) + G_2(\eta)) d\Gamma(\eta) - \\ & - \frac{1}{4\pi R G} \int_{\Gamma} \frac{(h(\eta) - h(\xi))^2}{\ell^3(\xi \cdot \eta)} \Delta g(\eta) d\Gamma(\eta) \end{aligned} \quad (1.1)$$

$$\begin{aligned} v(\xi) = & - \frac{1}{4\pi G} \int_{\Gamma} \text{Grad}_{\xi} \text{St}(\xi \cdot \eta) (\Delta g(\eta) + G_1(\eta) + G_2(\eta)) d\Gamma(\eta) + \\ & + \frac{1}{4\pi R^2 G} \int_{\Gamma} (h(\eta) - h(\xi))^2 \text{Grad}_{\xi} \frac{1}{\ell^3(\xi \cdot \eta)} \Delta g(\eta) d\Gamma(\eta) - \\ & - \frac{\Delta g(\xi) + G_1(\xi)}{R G} \text{Grad } h(\xi) \end{aligned} \quad (1.2)$$

where

$$G_1(\xi) = \frac{1}{2\pi R} \int_{\Gamma} \frac{h(\eta) - h(\xi)}{\ell^3(\xi \cdot \eta)} \Delta g(\eta) d\Gamma(\eta) \quad (1.3)$$

$$\begin{aligned} G_2(\xi) = & \frac{1}{2\pi R} \int_{\Gamma} \frac{h(\eta) - h(\xi)}{\ell^3(\xi \cdot \eta)} G_1(\eta) d\Gamma(\eta) + \\ & + \frac{1}{R^2} \Delta g(\xi) \text{Grad } h(\xi)^2 \end{aligned} \quad (1.4)$$

## Explanation of notation:

$\Gamma$	=	unit sphere
$\xi, \eta$	=	unit vectors, denoting points on $\Gamma$ . They play the role of spherical coordinates:  $\xi = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)$ astronomical coordinates are obtained by putting $\varphi = \frac{\pi}{2} - \theta$
$\xi \cdot \eta$	=	inner product of $\xi, \eta$ : frequently denoted by $\cos \psi$

$l(\xi \cdot \eta)$  = distance between  $\xi$  and  $\eta$ :

$$l(\xi \cdot \eta) = l(\cos \psi) = 2 \sin \frac{\psi}{2}$$

$St(\xi \cdot \eta)$  = Stokes' function

$$St(\xi \cdot \eta) = St(\cos \psi) = \frac{1}{\sin \frac{\psi}{2}} + \dots$$

(See Appendix B, equ.(B.24))

$R$  = mean earth radius

$G$  = mean gravity value

$\Delta g(\xi)$  = gravity anomaly referring to a point on the earth's surface with astronomical coordinates implied by  $\xi$ .

$h(\xi)$  = height above sea level in  $\xi$ .

$a(\xi)$  = height anomaly at  $\xi$  in the sense of Molodensky et al (1962); in the literature usually denoted by  $\zeta$ . Also the undulation of the quasi-geoid.

$v(\xi)$  = deflection of the vertical; viewed as a surface tangential vector to  $\Gamma$ . In the literature usually denoted by its components  $\xi, \eta$  in a localized coordinate system.

$\text{Grad}_\Gamma f(\xi)$  = denotes the surface gradient of a function  $f(\xi)$  defined on  $\Gamma$ . The surface gradient is a vector tangential to  $\Gamma$  in  $\xi$  and pointing into the direction of maximal increase of  $f(\xi)$ . The length of the vector equals the rate of the increase.

A derivation and discussion of the physical meaning of these formulas will not be given here. They will be used as a starting point for this study except for one thing: During the derivation of the formulas "planar approximation" has been used. This means that in deriving a formula for a certain quantity a relative error of the order  $h/R$  has been tolerated. We shall do the same whenever there will be a need for modifying some of the formulas.

The accuracy which shall be aimed at is about three digits in the quantities  $a(\xi)$  and  $v(\xi)$ . This does not necessarily mean that this accuracy can be guaranteed throughout. It means, rather, that we do not worry about approximations which create errors of less than three digits in  $a(\xi)$  and  $v(\xi)$ .

## 2. THE INTEGRO-DIFFERENTIAL CHARACTER OF THE FORMULAS

The formulas (1.1) to (1.4) are frequently referred to as integral formulas. Though it is recognized that some of the involved integrals are singular, the evaluation of the formulas is usually viewed as a problem of numerical integration.

Let us first have a look at the integral (1.3) serving the evaluation of  $G_1(\xi)$ . This integral is in an abbreviated way written as

$$G_1 = \frac{1}{2\pi R} \int_{\Gamma} \frac{\bar{h} - h}{\ell^3} \bar{\Delta g} d\Gamma \quad (2.1)$$

Although we shall use for the actual evaluation of this integral another transformation; we write (2.1) now as

$$G_1 = \frac{1}{2\pi R} \int_{\Gamma} \frac{\bar{h} \bar{\Delta g} - h \Delta g}{\ell^3} d\Gamma - \frac{1}{2\pi R} h \int_{\Gamma} \frac{\bar{\Delta g} - \Delta g}{\ell^3} d\Gamma \quad (2.2)$$

Here we have two integrals of the type

$$y = \frac{1}{2\pi} \int_{\Gamma} \frac{\bar{x} - x}{\ell^3} d\Gamma \quad (2.3)$$

The spherical harmonics equivalent of this is

$$y_{nm} = -n x_{nm} \quad (2.4)$$

This shows an amplification of the higher harmonics of the same type as it occurs in a differentiation. As outlined in Meissl (1971) a transformation of the type (2.4) can be viewed as a transformation from a space  $H^{k+1}$  into a space  $H^k$  where the members of  $H^{k+1}$  possess certain generalized derivatives up to order  $k+1$  whereas those of  $H^k$  possess only such derivatives up to order  $k$ .

Thus (2.3) acts similar to a differentiation procedure which always makes things more rough than they have been before.

During the evaluation of  $G_2$  such a roughing effect occurs twice. Stokes' formula has an opposite effect. It transforms from  $H^k$  into  $H^{k+1}$ . It, so to speak, decreases the roughing procedure by one step. Nevertheless in summary it turns out that  $a(\xi)$  depends in some way on derivatives of  $h$  and  $\Delta g$ . Whereas the mean contribution to  $a(\xi)$ ; namely, :

$$\frac{R}{4\pi G} \int_{\Gamma} \text{St } \overline{\Delta g} \, d\Gamma,$$

is smooth (one step smoother than  $\Delta g$ ), the correctional terms are rougher. This has serious consequences on any theoretical discussion of the Molodensky approach which will, however, not be undertaken here.

The consequences which have to be dealt with here are that there will be a need to evaluate in a consistent way derivatives of functions which originally are given only in discrete locations. We shall briefly outline now what we mean by consistent. Take for example Green's second identity

$$\int_B (f \text{ Lap } g - g \text{ Lap } f) \, d\Gamma = \int_{\partial B} f (\text{Grad } g, \nu) - g (\text{Grad } f, \nu) \, d\partial B \quad \dots (2.5)$$

Lap denotes the surface Laplacean operator with respect to  $\Gamma$ .  $B$  is a sub-area of  $\Gamma$  with boundary  $\partial B$ . The unit vector  $\nu$  is tangential to  $\Gamma$  and normal to  $\partial B$ . ( $\nu$  is directed outward of  $B$ ).

If now the functions  $f$  and  $g$  are given only at certain discrete locations then no integral in (2.5) can be evaluated. If the involved derivatives are computed by some crude and inaccurate numerical interpolation and differentiation procedure then (2.5) could possibly not be verified. We shall, however, use transformations of our integrals which are based on (2.5). Therefore what is clearly needed is a procedure to evaluate the derivatives of the functions in a way that (2.5) holds to a satisfactory degree of accuracy. We have to interpolate the discrete values in a way that the function is continuously differentiable up to the second order. There are many ways to do this and the result is, of course, not unique. We reject trigonometric

or polynomial interpolation in the classical sense and choose instead the spline function approach. The latter yields generally a smoother surface because local disturbances have hardly any influences on distant areas. This is more fully explained in Appendix A. The way data are assumed to be given, i.e. what the locations of the discrete arguments are, is explained in section 4. For the immediately following sections, it suffices to know that the functions and their derivatives up to second order are defined everywhere on  $\Gamma$ .

### 3. REGULARIZATION OF THE INTEGRALS

A singular integral can be transformed into a regular one if the integrand is sufficiently smooth. This may for the first be illustrated by the following simple example.

#### 3.1. A one-dimensional example

Take:

$$g(x) = \int_{-1}^{+1} \frac{\varphi(x, y)}{(x - y)^2} dy \quad (3.1)$$

We assume that  $\varphi(x, y)$  is twice continuously differentiable and that

$$\varphi(x, x) = 0 \quad (3.2)$$

The integral (3.1) does not exist if the integrand is replaced by its absolute value.

The integral is therefore singular. Writing instead of (3.1)

$$g_{\epsilon}(x) = \int_{-1}^{x-\epsilon} + \int_{x+\epsilon}^{+1} = (\epsilon) \int_{-1}^{+1}, \text{ say,}$$

the singular integral  $g(x)$  is defined by  $\lim_{\epsilon \rightarrow 0} g_{\epsilon}(x)$ .

Applying partial integration toward  $g_{\epsilon}(x)$  yields

$$g_{\epsilon}(x) = \left. \frac{\varphi(x, y)}{x - y} \right|_{-1}^{x-\epsilon} + \left. \frac{\varphi(x, y)}{x - y} \right|_{x+\epsilon}^{+1} - (\epsilon) \int_{-1}^{+1} \frac{\varphi_y(x, y)}{x - y} dy$$

Applying partial integration a second time gives

$$g_{\epsilon}(x) = \left. \frac{\varphi(x, y)}{x - y} \right|_{-1}^{x-\epsilon} + \left. \frac{\varphi(x, y)}{x - y} \right|_{x+\epsilon}^{+1}$$

....(con't)

$$\begin{aligned}
& + \varphi_y(x, y) \ln|x - y| \Big|_{-1}^{x-\epsilon} + \varphi_y(x, y) \ln|x - y| \Big|_{x+\epsilon}^1 \\
& - (\epsilon) \int_{-1}^1 \varphi_{yy}(x, y) \ln|x - y| dy
\end{aligned}$$

If we now let  $\epsilon \rightarrow 0$  then the various poles cancel out neatly and we obtain

$$\begin{aligned}
g(x) = & - \frac{\varphi(x, -1)}{1+x} - \frac{\varphi(x, 1)}{1-x} - \varphi_y(x, -1) \ln|1+x| + \varphi_y(x, 1) \ln|1-x| \\
& + 1 \\
& - \int_{-1}^1 \varphi_{yy}(x, y) \ln|x - y| dy
\end{aligned} \tag{3.3}$$

The integral is now completely regular, which means that it exists even if the integrand is replaced by its absolute value.

### 3.2 Regularization of M-type integrals

Let's go now to the unit sphere  $\Gamma$ . Assume an integral of the form

$$g(\xi) = \int_{\Gamma} \frac{\varphi(\xi, \eta)}{\ell^3(\xi, \eta)} d\Gamma(\eta) \tag{3.4}$$

with

$$\varphi(\xi, \xi) = 0 \tag{3.5}$$

and  $\varphi(\xi, \eta)$  at least twice continuously differentiable. We call this a type M-integral since the first order correction term in Molodensky's formula is of this type, e.g. equation (1.3). The singular integral is defined after excluding a very small circular cap of half opening angle  $\epsilon$  around  $\xi$  and then letting  $\epsilon \rightarrow 0$ .

The equivalent of the above two-fold partial integration is in the two (or more) dimensional case, nothing but Green's second formula (2.5).

We transform the integral not over the whole sphere  $\Gamma$  but only over a certain

spherical cap  $C$  surrounding  $\xi$  and having half-opening angle  $\psi_0$ .

It is shown in Appendix C that

$$\begin{aligned}
 g_C(\xi) &= \int_C \frac{\varphi(\xi, \eta)}{\ell^3(\xi, \eta)} d\Gamma(\eta) = \\
 &= \int_C \left\{ \frac{1}{\ell(\xi, \eta)} - \frac{1}{\ell(\cos \psi_0)} \right\} \text{Lap}_\eta \varphi(\xi, \eta) d\Gamma(\eta) - \\
 &\quad - \frac{\cos \frac{\psi_0}{2}}{\ell^2(\cos \psi_0)} \oint_{\partial C} \varphi(\xi, \eta) d\partial C(\eta) \\
 &\quad - \frac{1}{4} \int_C \frac{\varphi(\xi, \eta)}{\ell(\xi, \eta)} d\Gamma(\eta) \tag{3.6}
 \end{aligned}$$

The formula becomes simpler if the cap is extended over the whole sphere; in that case the line integral vanishes. However it is not advisable to do so. The term  $1/\ell^3(\xi, \eta)$  in (3.4) tapers off quickly for larger distances. This is not so in (3.6) where only terms like  $1/\ell(\xi, \eta)$  occur. Therefore for distant areas (3.4) offers greater advantage. The term  $\text{Lap}_\eta \varphi(\xi, \eta)$  in (3.6) certainly tends to fluctuate. If the integral is extended over too large a cap then the positive and negative contributions of  $\text{Lap}_\eta \varphi(\xi, \eta)$  certainly cancel to some extent. Therefore the cap should not be too large allowing only a few positive and negative extremes of  $\text{Lap}_\eta \varphi(\xi, \eta)$  within its domain.

In all numerical approaches toward the singular integrals of Physical Geodesy which are known to me the integration according to (3.4) is carried out very close to the point  $\xi$  of singularity. For a very small remaining cap some rather crude procedure is used which involves some kind of numerical differentiation. This does not seem to be a very efficient procedure. A singular integral yields infinity if the integrand is replaced by its absolute value. Therefore if only a very small cap around the singularity is excluded, there are necessarily large positive and negative contributions which nearly cancel. This is numerically disadvantageous even if a precise

analytical expression for the integrand is available. An analytical representation for the integrand appears to be a necessity in any case. If one is available then derivatives may be formed. Formula (3.6) can then be evaluated without any numerical trouble resulting from mutual contributions of large quantities of opposite sign. The cap shall not be too small but also not too large as we have pointed out earlier.

M-type integrals appear in our formulas of section 1 multiplied by a factor  $\frac{1}{R}$ . In that case the last integral in (3.6) is a quantity which is negligible in planar approximation. (Moritz, 1969).

In most cases, the cap size will be sufficiently small to replace (3.6) by a plane integral.

In that case we proceed in two steps. We assume that  $\frac{1}{R} g_C(\xi)$  has to be evaluated. In doing this we replace the integration over the unit sphere by one over a sphere with radius  $R$ . (3.6) becomes then after neglecting the last integral

$$\begin{aligned} \frac{1}{R} g_C(\xi) &= \int_C \frac{\varphi(\xi, \eta)}{|R\xi - R\eta|^3} R^2 d\Gamma(\eta) \\ &= \int_C \left\{ \frac{1}{|R\xi - R\eta|} - \frac{1}{R \ell(\cos \psi_0)} \right\} \frac{1}{R^2} \text{Lap}_\eta \varphi(\xi, \eta) R^2 d\Gamma(\eta) \\ &\quad - \frac{\cos \frac{\psi_0}{2}}{(R \ell(\cos \psi_0))^2} \oint_{\partial C} \varphi(\xi, \eta) R d\partial C(\eta) \end{aligned} \quad (3.7)$$

Now we make the transition to the plane replacing  $R\xi$  by  $x = (x_1, x_2)^T$ ,  $R\eta$  by  $y = (y_1, y_2)^T$  and denoting by  $\rho$  the radius of the cap. The expression  $\frac{1}{R^2} \text{Lap}_\eta \varphi(\xi, \eta)$  goes over into the ordinary plane Laplacian  $\text{lap}_y \varphi(x, y) = \varphi_{y_1 y_1} + \varphi_{y_2 y_2}$ . Hence:

$$\begin{aligned}
\frac{1}{R} g_C(x) &= \int_{|y| \leq \rho} \frac{\varphi(x, y)}{|x - y|^3} dy = \\
&= \int_{|y| \leq \rho} \left\{ \frac{1}{|x - y|} - \frac{1}{\rho} \right\} \text{lap}_y \varphi(x, y) dy \\
&\quad - \frac{1}{\rho} \int_{\alpha=0}^{2\pi} \varphi(x, x + \rho \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix}) d\alpha \quad (3.8)
\end{aligned}$$

It will be clear from later sections, that we shall have a polynomial representation for the function  $\varphi(x, y)$ . However, this polynomial representation will vary from mesh to mesh of a rectangular grid. It may therefore be desirable to have a cap boundary which is not circular but rectangular. Call this rectangular area  $Q$ . Then (3.8) becomes:

$$\begin{aligned}
\frac{1}{R} g_Q(x) &= \int_Q \frac{\varphi(x, y)}{|x - y|^3} dy = \\
&= \int_Q \frac{1}{|x - y|} \text{lap}_y \varphi(x, y) dy \\
&\quad + \oint_{\partial Q} \varphi(x, y) \left( \text{grad}_y \frac{1}{|x - y|}, v(y) \right) d\partial Q(y) \\
&\quad - \int_{\partial Q} \frac{1}{|x - y|} \left( \text{grad}_y \varphi(x, y), v(y) \right) d\partial Q(y) \quad (3.9)
\end{aligned}$$

These formulas could also be verified by Green's second formula for the plane considering that

$$\text{lap}_y \frac{1}{|x - y|} = \frac{1}{|x - y|^3} \quad (3.10)$$

$v$  is the outer normal to the boundary of  $Q$  and  $\text{grad}$  denotes the ordinary gradient in the plane. Thus the second equal sign in (3.8) and (3.9) is exact. Only the

equality with the integral over the portions of the sphere is approximate.

### 3.3 Regularization of Vening-Meinesz' formula

Vening Meinesz' formula

$$v(\xi) = - \frac{1}{4\pi G} \int_{\Gamma} \text{Grad}_{\xi} \text{St}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (3.11)$$

constitutes also a singular integral since it does not exist after replacing the integrand by its absolute value. Regularization may take place in a similar way. Again, only the contribution  $v_C(\xi)$  of a circular cap  $C$  centered at  $\xi$  and having spherical radius  $\psi_0$  is considered. Farther outside the original form (3.11) is quite suitable.

The regularization procedure is outlined in Appendix D. It results in:

$$\begin{aligned} v_C(\xi) = & \frac{1}{2\pi G} \int_C \sigma(\xi, \eta) \text{Lap} \Delta g(\eta) d\Gamma(\eta) - \\ & - \frac{1}{2\pi G} \oint_{\partial C} \sigma(\xi, \eta) (\text{Grad} \Delta g(\eta), \nu(\eta)) d\partial C(\eta) + \\ & + \frac{1}{2\pi G} \int_C \frac{1 - \cos \frac{3\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) \Delta g(\eta) d\Gamma(\eta) - \\ & - \frac{1}{4\pi G} \int_C \text{Grad}_{\xi} R(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \end{aligned} \quad (3.12)$$

Thereby we have split:

$$\text{St}(\xi \cdot \eta) = \frac{2}{l(\xi \cdot \eta)} + R(\xi \cdot \eta) \quad (3.13)$$

The unit vector  $\sigma(\xi, \eta)$  is tangential to  $\Gamma$  in the point  $\xi$  and lies in the same plane as  $\xi$  and  $\eta$ .  $R(\xi \cdot \eta)$  causes, even after differentiation, no singularity. The singularity in Vening Meinesz' formula comes in through the term  $2/l(\xi \cdot \eta)$  in (3.13) and has been removed in the above formula. In Appendix D there is also a listing of the above formula in the usual notation which is based on a localized coordinate system

(in which  $\sigma(\xi \cdot \eta)$  carries over into  $(\frac{\cos \alpha}{\sin \alpha})$ . Equation (D.9a).

Again, formula (3.12) would become simpler if  $C$  would be replaced by  $\Gamma$ . The line integrals would vanish. The numerical Usefulness, however, would decrease for reasons similar to those given in section 3.2.

#### 3.4. Some further manipulations

There is still a singular integral in the deflection formula (1.2). It is

$$q(\xi) = \frac{1}{4\pi R^2 G} \int_C (h(\eta) - h(\xi))^2 \text{Grad}_{\xi} \frac{1}{\ell^3(\xi \cdot \eta)} \Delta g(\eta) d\Gamma(\eta)$$

This term is combined with a certain other contribution toward (1.2) and regularized as outlined in Appendix E. A summary of all regularizations is given in the following subsection.

### 3.5 Summary of regularization

#### 3.5.1 The Quantity $G_1(\xi)$ .

For the area outside the cap  $C$  use the original formula (1.3). For the cap-interior use:

$$G_{1C}(\xi) = \frac{1}{2\pi R} \int_C \left\{ \frac{1}{\ell(\xi\eta)} - \frac{1}{\ell(\cos \psi_0)} \right\} \text{Lap}_\eta [(h(\eta) - h(\xi)) \Delta g(\eta)] d\Gamma(\eta) \\ - \frac{1}{2\pi R} \frac{\cos \frac{\psi_0}{2}}{\ell^2(\cos \psi_0)} \oint_{\partial C} (h(\eta) - h(\xi)) \Delta g(\eta) d\partial C(\eta) \quad (3.18)$$

This formula follows from (3.6) after neglection of the last integral (planar approximation).

#### 3.5.2 The quantity $G_2(\xi)$ .

Split

$$G_2(\xi) = \bar{G}_2(\xi) + \bar{\bar{G}}_2(\xi) \quad (3.19)$$

$$\bar{G}_2(\xi) = \frac{1}{2\pi R} \int_\Gamma \frac{h(\eta) - h(\xi)}{\ell^3(\xi\eta)} G_1(\eta) d\Gamma(\eta) \quad (3.20)$$

$$\bar{\bar{G}}_2(\xi) = \frac{1}{R^2} \Delta g(\xi) |\text{Grad } h(\xi)|^2 \quad (3.21)$$

For  $\bar{G}_{2C}$  use (3.18) with  $\Delta g(\eta)$  replaced by  $G_1(\eta)$ . The evaluation of  $\bar{\bar{G}}_2(\xi)$  is immediate.

#### 3.5.3 Formula (1.1) For The Height Anomaly $a(\xi)$ .

No further regularization is necessary.

### 3.5.4. The Deflection - Formula.

For the area outside the cap  $C$  the original formula (1.2) will be used.

For the cap interior the following will be used (cf. (3.13)):

$$\begin{aligned}
 v_C(\xi) = & \frac{1}{2\pi G} \int_C \sigma(\xi, \eta) \text{Lap}[\Delta g(\eta) + G_1(\eta) + \bar{G}_2(\eta)] d\Gamma(\eta) - \\
 & - \frac{1}{2\pi G} \oint_{\partial C} \sigma(\xi, \eta) (\text{Grad}[\Delta g(\eta) + G_1(\eta) + \bar{G}_2(\eta)], \nu(\eta)) d\partial C(\eta) + \\
 & + \frac{1}{2\pi G} \int_C \frac{1 - \cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) [\Delta g(\eta) + G_1(\eta) + \bar{G}_2(\eta)] d\Gamma(\eta) - \\
 & - \frac{1}{4\pi G} \int_C \text{Grad}_\xi R(\xi, \eta) [\Delta g(\eta) + G_1(\eta) + \bar{G}_2(\eta) + \bar{\bar{G}}_2(\eta)] d\Gamma(\eta) + \\
 & + \frac{1}{4\pi R^2 G} \int_C \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) \left\{ \text{Lap}_\eta [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \right. \\
 & \quad \left. - 2|\text{Grad } h(\eta)|^2 \Delta g(\eta) \right\} d\Gamma(\eta) \\
 & + \frac{1}{2\pi R^2 G} \int_C \frac{1 - \cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) |\text{Grad } h(\eta)|^2 \Delta g(\eta) d\Gamma(\eta) \\
 & - \frac{1}{4\pi R^2 G} \oint_{\partial C} \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) (\text{Grad}_\eta [(h(\eta) - h(\xi))^2 \Delta g(\eta), \\
 & \quad , \nu(\eta)) d\partial C(\eta) \\
 & - \frac{1}{2\pi R^2 G} \oint_{\partial C} [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \frac{\cos \psi}{\sin^3 \psi} \sigma(\xi, \eta) d\partial C(\eta) \\
 & - \frac{1}{4\pi R^2 G} \int_C [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \left\{ \frac{3(1 - \cos^5 \frac{\psi}{2})}{\sin^4 \psi} \right. \\
 & \quad \left. - \frac{2}{\sin^2 \psi} \right\} \sigma(\xi, \eta) d\Gamma(\eta) \quad (3.22)
 \end{aligned}$$

#### 4. ANALYTIC REPRESENTATION OF FUNCTIONS

It has been stated repeatedly that our regularization procedures are based on the availability of an analytical representation of the various involved functions like terrain heights, gravity anomalies, and other functions derived from them. Data libraries contain information about these functions either in discrete form or in block average form. Neither of these representations is suited for our purposes. Block averages produce step functions with constant values within each block and jumps at the block boundary. Such a function is not analytical nor can it be reconstructed from its first and second derivatives which are zero almost everywhere. For our regularization procedure it is, however, essential that the functions can be reconstructed from these derivatives (plus some additional information like the function values on a boundary line).

If gravity anomalies and terrain heights are given at discrete values, an interpolation procedure has to be used. For an irregular pattern of sample points in two dimensions this can be a quite laborous task. Usually a linear procedure is used leading to an expression

$$p(x,y) = \sum_{i=1}^N z_i p_i(x,y) \quad (4.1)$$

$z_i$  are the values of the functions for the arguments  $x_i, y_i$ .  $p(x,y)$  is the interpolating function. The functions  $p_i(x,y)$  depend on only the arguments  $x_i, y_i$  and, of course, on the chosen interpolation procedure. They do not depend on the  $z_i$ .

Examples are: polynomial interpolation, trigonometric interpolation, linear prediction. If the function  $p_i(x,y)$  is differentiable, then  $p(x,y)$  is differentiable. This generally solves our problem. The question is how to choose a computationally feasible interpolation procedure.

For an irregular pattern of points there is hardly a chance to get simple expressions for  $p(x,y)$ . Therefore a rectangular grid of points  $(x_i, y_j)$  points must be achieved. This can be done for example by a prediction procedure. The interpolation within the rectangular grid is according to the author's opinion, in the best way

accomplished by spline interpolation. Spline interpolation yields a polynomial expression for the function within each mesh  $x_i \leq x \leq x_{i+1}$ ,  $y_j \leq y \leq y_{j+1}$  of the grid. The polynomial expression varies from mesh to mesh. However, continuity and continuous differentiability up to any desired order can be maintained at the mesh boundaries. Advantages of spline interpolation over other methods are outlined in Appendix A. Loosely speaking, spline interpolation keeps the derivatives of the interpolating function as small as possible, thus avoiding fluctuations and oscillations as they are frequently encountered in other methods. Another advantage is that the polynomial representation within a certain mesh depends to a large extent only on the function values  $z_{ij}$  in the near neighborhood of the mesh. The dependency on values farther outside is negligibly small. This is also not the case with other methods where a change of one value  $z_{ij}$  may cause changes in the interpolating function over a wide area.

In Appendix A computational procedures are outlined to obtain the interpolating bicubic spline functions. Asymptotic formulas have been used which are simple and quite appropriate for our purposes.

Spline interpolation can also be applied toward the block averages. It is only necessary to view the block averages in a different way. They should not be viewed as step functions. Instead a block average shall be representative only for its midpoint. For any other point the appropriate value would be an average over a block centered at that point. In other words we imagine a smoothed version of our function which is obtained by moving averages over blocks. Unfortunately we have values of this smoothed version only at discrete values, i.e. the midpoints of the original blocks. For the other points, an interpolation procedure has to be used. This can certainly be spline interpolation since the midpoints already form a rectangular grid.

Speaking of a rectangular grid does not mean that the lines must be equidistant on the unit sphere  $\Gamma$ . The grid lines can certainly be represented by lines of constant latitude and longitude. In evaluating the various differential operators like Grad and Lap the appropriate formulas have to be used.

Spline interpolation will, of course, only be used for the small cap areas in which regularization takes place. For the outer zones the original averages or even cruder representations will be sufficient. Here one benefit of spline interpolation becomes effective. The polynomial representations for a mesh is practically dependent only on the function values in a rather small neighborhood of the mesh. This avoids the necessity of performing an a priori interpolation for larger areas where continuity problems arise at the connecting boundaries.

## 5. TRUNCATION

The procedure for evaluating the various integrals will be the following. For a very small cap with half opening angle  $\tilde{\psi}_0$  the regularized formulas will be used in case the integral is singular. The radius of this cap will be a few kilometers or even less. This corresponds to a  $\tilde{\psi}_0$  of about 1'. Outside this cap essentially the original form of the integrals will be used. It is, however, certainly not necessary to carry out the integration with the same accuracy all over the sphere. Some integrals can be completely truncated at a certain distance. Some have to be extended over all of  $\Gamma$ . However, going farther out more smoothed versions of the functions  $h$ ,  $\Delta g$ , ... can be used. This reduces the computation time.

There exists a theory on truncation which goes back to Molodensky. See Molodensky et al. (1962). De Witte and others have done further work. In Appendix B I have reviewed and, as I think, somewhat extended this theory.

New insight has been gained into the nature of the truncation error. Take for example Stokes' formula. De Witte's work suggests that truncation is most favorable at the zeros of Stokes' function. However a zero can be placed anywhere by simply adding a constant to Stokes' function. Doing this and truncating at such an enforced zero produces a genuine truncation error which is comparable in size to that after truncation at a zero of the unmodified Stokes' function. The only difference is that the new truncation error is superimposed by a constant which is the same for all points of the sphere. Such a constant does not matter in Stokes' formula. The geoid has to be scaled later on anyway.

We have also investigated the effect of replacing the abrupt truncation at a certain angle  $\psi_0$  by a smoother procedure. It consists of letting the kernel taper off to zero over a certain interval  $\psi_1 \leq \psi \leq \psi_0$ . This brings along a very beneficial effect in dampening the higher harmonics of the truncation error. The situation is similar to filter design theory. Filter functions with discontinuities tend to cause errors with considerable high frequent portions. Smoothing out the discontinuities removes these high frequent errors to some extent.

In the following we discuss truncation procedures for the various integrals. Alternatives to the proposed strategies can be obtained from Appendix B.

### 5.1 The integrals of the correctional type

$G_1$  is a correction of  $\Delta g$  as it is seen from (1.1) and (1.2). Therefore a contribution toward  $G_1$  can be neglected if it amounts to less than the measuring accuracy of  $\Delta g$ . This accuracy may be assumed as one part in 1000. Due to this and also due to the fact that the function  $1/\ell^3(\xi \cdot \eta)$  tapers off quickly truncation can take place at a short distance. Bursa (1965) proposes truncation beyond a distance of about 80 km. This corresponds to  $\psi_0 \approx 0.7^\circ$ . A global statement is difficult to make but some rough estimates based on the truncation theory of Appendix B indicate that even in the case of our second order approach not much more is needed.  $\psi_0 \approx 2^\circ$ , should be sufficient in all cases.

The same can be said about all other integrals of this correctional type. These are: The integral in  $G_2$ , the second integral in either of (1.1) and (1.2).

### 5.2 Stokes' integral

Whereas nothing new has been said in section 5.1, it is hoped that the following discussions on Stokes' and Vening Meinesz integral bear some interest.

We propose the following. Split off the spherical harmonic components up to and including degree 12 from Stokes' kernel:

$$St(\xi \cdot \eta) = \sum_{n=2}^{12} \frac{2n+1}{n-1} P_n(\xi \cdot \eta) + St_{12}(\xi \cdot \eta) \quad (5.1)$$

The contribution of the first part will not be discussed further. It could be taken from satellite solutions. The residual kernel  $St_{12}(\cos \psi)$  has at least 13 zeros in  $0 \leq \psi \leq 180$ . (Meissl (1971)). The zeros are approximately given by

$$\psi = 5.02^\circ, 17.99^\circ, 31.58^\circ, 45.29^\circ, 59.06^\circ, 72.85^\circ, 86.65^\circ, 100.5^\circ, \\ 114.3^\circ, 128.1^\circ, 141.9^\circ, 155.7^\circ, 169.4^\circ.$$

These zeros are good locations for truncation, or better for a transition to a heavier smoothed version of  $\Delta g$  in the residual Stokes' formula:

$$N_{12}(\xi) = \frac{R}{4\pi G} \int_{\Gamma} St_{12}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (5.2)$$

In (1.1)  $\Delta g$  is replaced by  $\Delta g + G_1 + G_2$ . However for the sake of simplicity we discuss the (residual) Stokes formula in the form (5.2).

Now we integrate only up to  $\psi_0 = 5.02$  with full accuracy. Beyond  $\psi = 5.02$  we replace  $\Delta g(\xi)$  by a smoothed version  $\Delta g^{(1)}(\xi)$  which is obtained by averaging  $\Delta g(\xi)$  over a circular disk of half opening angle  $\alpha_0$  centered at  $\xi$ . The spherical harmonic expansion of the resulting error  $\Delta N_{12}^{(0)}(\xi)$  in  $N_{12}(\xi)$  is given by (B.82) and (B.45)

$$(\Delta N_{12}^{(0)})_{nm} = \frac{R}{G} \frac{Q_n^{(0)}}{2} (1 - \beta_n^{(0)}) \Delta g_{nm} \quad (5.3)$$

$Q_n^{(0)}$  are the Molodensky coefficients referring to the residual Stokes' kernel  $St_{12}(\xi \cdot \eta)$  and a truncation angle equal to  $\psi_0$ . The  $Q_n^{(0)}$  can be computed from (B.46) if  $St(\xi \cdot \eta)$  is replaced by  $St_{12}(\xi \cdot \eta)$  and the summation over  $m$  is extended to  $N=12$ . The  $\beta_n^{(0)}$  are the eigen values of the smoothing operator over the  $\alpha_0$ -disks. The  $\beta_n^{(0)}$  are given by (B.83) or, for not too large  $n$ , approximately by (B.86).

Let us use the smoothed version  $\Delta g^{(1)}$  only up to  $\psi_1 = 17.99$ . Outside the  $\psi_1$  cap a still heavier smoothed version  $\Delta g^{(2)}(\xi)$  will be employed.  $\Delta g^{(2)}$  is obtained by averaging  $\Delta g$  over disks of half opening angle  $\alpha_1$ ,  $\alpha_1 > \alpha_0$ . The total error would then be given by

$$(\Delta N_{12}^{(1)})_{nm} = (\Delta N_{12}^{(0)})_{nm} + \frac{R}{G} \frac{Q_n^{(1)}}{2} (\beta_n^{(0)} - \beta_{nm}^{(1)}) \Delta g_{nm} \quad (5.4)$$

$Q_n^{(1)}$ : Molodensky coefficients for truncating  $St_{12}(\cos \psi)$  at  $\psi = \psi_1$ .

$\beta_n^{(1)}$ : eigen values of the averaging operator over the  $\alpha_1$ -disks.

This procedure is immediately generalized to a succession of zones. If  $\psi_i$

is the last truncation angle then the associated error in  $N_{12}$  is  $\Delta N_{12}^{(i)}$ . The following recursion holds:

$$(\Delta N_{12}^{(i)})_{nm} = (\Delta N_{12}^{(i-1)})_{nm} + \frac{R}{G} \frac{Q_n^{(i)}}{2} (\beta_n^{(i-1)} - \beta_n^{(i)}) \Delta g_{nm} \quad (5.5)$$

The general form of the error is

$$(\Delta N_{12}^{(i)})_{nm} = \frac{R}{G} q_n^{(i)} \Delta g_{nm} \quad (5.6)$$

If  $\sigma^2(\Delta g)$  are the degree variances of  $\Delta g$  then the mean square error  $\sigma^2(\Delta N_{12}^{(i)})$  is given by

$$\sigma^2(\Delta N_{12}^{(i)}) = \frac{R^2}{G^2} \sum_{n > 12} (q_n^{(i)})^2 \sigma_n^2(\Delta g) \quad (5.7)$$

See for example Meissl (1971) chapter 4, esp. equ. (4.11) there.

We adopt the following degree variances for  $\Delta g$  (they are taken from Gaposchkin-Lambeck (1970). ).

n	$\sigma_n^2(\Delta g)$ in $\text{mg}^2$
2	7.4
3	33.0
4	20.0
5	17.8
6	15.7
7	15.5
8	6.7
9	12.7
10	12.9
11	12.2
12	5.1

Occasionally we shall need higher degree variances too. Since we need them only for rough estimates we adopt the following model for the  $\sigma_n^2(\Delta g)$ ,  $n > 12$ :

$$\sigma_n^2(\Delta g) = \frac{3 \cdot 10^{11}}{(n + 445)^4} \quad (5.8)$$

Summing all degree variances should give a value close to  $1201 \text{ mg}^2$ . (Kaula (1959).

We have from the above table

$$\sum_{n=2}^{12} \sigma_n^2(\Delta g) = 159 \quad (5.9)$$

and from (5.8)

$$\sum_{n > 12} \sigma_n^2(\Delta g) = 1042 \quad (5.10)$$

which gives together 1201. Besides that we have for  $n=13$ , a value  $\sigma_{13}^2(\Delta g) \approx 7 \text{ mg}^2$  which fits into the picture of the above table.

The variance of the geoidal undulations is then

$$\sigma^2(N) = \frac{R^2}{G^2} \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} \sigma_n^2(\Delta g) = (30.2 \text{ m})^2 \quad (5.11)$$

This is the total variance. The variance of the residual part  $N_{12}$  is

$$\sigma^2(N_{12}) = \frac{R^2}{G^2} \sum_{n > 12} \frac{1}{(n-1)^2} \sigma_n^2(\Delta g) = (6.3 \text{ m})^2 \quad (5.12)$$

In Table 5-1 we have listed  $q_n^{(i)}$  quantities under various assumptions concerning the underlying  $\alpha_i$ 's.

Take for example an exact representation of  $\Delta g$  within  $\psi_0 = 5.02$  and moving averages over circular disks with half opening angle  $\alpha_0 = 0.56$  outside. This corresponds roughly to  $1^\circ \times 1^\circ$  blocks outside. The  $q_n^{(0)}$  are seen not to exceed 0.0001 for  $n > 12$ . This enables us to compute an error bound which is independent of the hypothesis (5.8). From (5.7) follows:

$$\sigma^2(\Delta N_{12}^{(0)}) = \frac{R^2}{G^2} \sum_{n > 12} (q_n^{(0)})^2 \sigma_n^2(\Delta g) \leq \frac{R^2}{G^2} 0.0001 \sum_{n > 12} \sigma_n^2(\Delta g)$$

From (5.10) follows now

$$\sigma(\Delta N_{12}^{(0)}) \leq \frac{R}{G} 0.0001 \sqrt{1042} \approx 0.02 \text{ m} \quad (5.13)$$

Under the hypothesis (5.8) even a smaller error can be estimated:  $\sigma(\Delta N_{12}^{(0)}) \approx 0.006 \text{ m}$ .

Truncating further at  $\psi_1 = 17.99$  and using moving averages based on  $\alpha_1 = 1.4$  outside (roughly  $2.5^\circ \times 2.5^\circ$  - blocks) produces an error not larger than .11 m. This again independent of hypothesis (5.8). This error would be obtained if all the power of  $\Delta g$  above degree 12 was concentrated at  $n = 13$ :  $\sigma_{13}^2(\Delta g) = 1042$ . This is certainly unrealistic. Under hypothesis (5.8) the estimate

$$\sigma(\Delta N_{12}^{(1)}) \approx 0.02 \text{ m} \quad (5.14)$$

holds.

Truncating further at  $\psi_2 = 31.58$  and using  $\alpha_2 = 2.8$  moving averages outside (roughly  $5^\circ \times 5^\circ$  - block averages) gives a total error

$$\sigma(\Delta N_{12}^{(2)}) \approx 0.05 \text{ m} \quad (5.15)$$

Hypothesis (5.8) has been used thereby.

### 5.3 Vening Meinesz's integral

Again we split according to (5.1) and assume that the contribution due to the harmonics up to and including  $n = 12$  can be taken care of otherwise. If truncation takes place at the zeros of  $St_{12}(\cos \psi)$ , then the theory of Appendix B, section B.6, tells us that the variance of the resulting error in

$$v_{12}(\xi) = \frac{-1}{4\pi G} \int_{\Gamma} \text{Grad}_{\xi} St_{12}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (5.16)$$

is given by

$$\sigma^2(\Delta v_{12}^{(i)}) = \frac{1}{G^2} \sum_{n>12} n(n+1) (q_n^{(i)})^2 \sigma_n^2(\Delta g) \quad (5.17)$$

See equ. (B.67a). The  $q_n^{(i)}$  are those of the previous section.

Table 5-1

$i$	$\psi_i$	$\alpha_i$
0	5.02	0.56
1	17.99	1.4
2	31.58	2.8

The  $q_n^{(1)}$  below are given in units of  $10^{-6}$ . If the  $n$ -column shows an interval rather than a number, then the extremum values within this interval are listed.

$n$	$q_n^{(0)}$	$q_n^{(1)}$	$q_n^{(2)}$
13	91	523	1905
14	90	446	1314
15	88	361	739
16	85	274	251
17	82	189	-100
18	79	110	-290
19	75	42	-328
20	70	-13	-250
21	66	-52	-108
22	60	-75	42
23	55	-82	151
24	49	-75	193
25	44	-57	165
26	38	-30	85
27	32	1	-15
28	26	32	-98
29	20	60	-138
30	14	82	-120
31- 35	-13	100	226
36- 40	-33	-91	-218
41- 50	-44	-118	-254
51- 75	-42	74	-142
76-100	34	64	98
101-150	-28	-55	-64
151-200	24	44	44
201-300	18	20	20
301-500	12	11	11

The overall variance of the deflection of the vertical is given by

$$\sigma^2(v) = \frac{1}{G^2} \sum_{n \geq 2} \frac{n(n+1)}{(n-1)^2} \sigma_n^2(\Delta g) \quad (5.18)$$

under the hypothesis (5.8) this leads to

$$\sigma(v) = 7.9'' \quad (5.19)$$

The residual deflections  $v_{12}$  have standard deviation obtainable by summing in (5.18) only over  $n > 12$ . This gives

$$\sigma(v_{12}) = 6.9'' \quad (5.20)$$

The truncation procedures outlined in the previous section produce errors which can be estimated in the following way.

$\psi_0 = 5.02, \alpha_0 = 0.56$	$\sigma(\Delta v_{12}^{(0)}) \leq 0.03''$	(without hypothesis (5.8) )
	$\approx 0.013''$	(under hypothesis (5.8) )
$\psi_1 = 17.99, \alpha_1 = 1.4$	$\sigma(\Delta v_{12}^{(1)}) \approx 0.02''$	(under hypothesis (5.8) )
$\psi_2 = 31.58, \alpha_2 = 2.8$	$\sigma(\Delta v_{12}^{(2)}) \approx 0.03''$	(under hypothesis (5.8) )
$\psi_3 = 45.29, \alpha_3 = 5.6$	$\sigma(\Delta v_{12}^{(3)}) \approx 0.07''$	(under hypothesis (5.8) )

## 6. A COMPUTATIONAL TEST

I am fully aware that any proposed numerical approach toward a mathematical problem should be thoroughly checked by test calculations. Unfortunately, time did not permit to do this on a large scale for the present study. In order to give some idea how the proposed methods can be converted into computer programs I shall deal in the following with bicubic spline interpolation and with the  $G_1$ -integral. We assume a plane grid which is quadratic and where the spacing between the lines is unity. We do not bother about a realistic scaling of the model. Let the grid consist of  $N^2$  points. Assume the functions  $h_{ij}$  and  $\Delta g_{ij}$  be defined for the grid points by

$$h_{ij} = \sum_{k=1}^N \frac{1}{i+2j+3k} \sin\left(\frac{ik\pi}{N}\right) \sin\left(\frac{jk\pi}{N}\right) \quad (6.1)$$

$$\Delta g_{ij} = \sum_{k=1}^N \frac{1}{i+2.5j+4k} \sin\left(\frac{ik\pi}{N}\right) \sin\left(\frac{jk\pi}{N}\right) \quad (6.2)$$

After this data generation we proceed to the bicubic spline interpolation. Let us outline this for  $h_{ij}$ . The procedure for  $\Delta g_{ij}$  is completely analogous. Compute according (A.7)

$$\sigma = -(2 - \sqrt{3}) \approx -.268$$

Now according (A.16) to (A.19) compute with  $\beta = 6$

$$m_{ij} = -3 \sum_{k=1}^6 \sigma^k (h_{i+k,j} - h_{i-k,j}) \quad \begin{array}{l} i = 7, \dots, N-6 \\ j = 1, \dots, N \end{array} \quad (6.3)$$

$$n_{ij} = -3 \sum_{\ell=1}^6 \sigma^\ell (h_{i,j+\ell} - h_{i,j-\ell}) \quad \begin{array}{l} i = 1, \dots, N \\ j = 7, \dots, N-6 \end{array} \quad (6.4)$$

$$p_{ij} = -3 \sum_{\ell=1}^6 \sigma^\ell (m_{i,j+\ell} - m_{i,j-\ell}) \quad \begin{array}{l} i = 7, \dots, N-6 \\ j = 7, \dots, N-6 \end{array} \quad (6.5)$$

A check could be made by using

$$p_{ij} = -3 \sum_{k=1}^6 \sigma^k (n_{i+k,j} - n_{i-k,j}) \quad \begin{array}{l} i = 7, \dots, N-6 \\ j = 7, \dots, N-6 \end{array} \quad (6.6)$$

If the agreement with (6.5) should not be satisfactory, then a  $\beta$  larger than 6 has to be chosen. In our test  $\beta = 6$  was sufficient. Now  $h_{ij}$  can be interpolated for any one of the meshes  $i \leq x \leq i+1$ ,  $j \leq y \leq j+1$  within the subgrid  $i = 7, \dots, N-7$ ,  $j = 7, \dots, N-7$ . The formulas are

$$h(x,y) = \sum_{k,\ell=0}^3 a_{k,\ell}^{(i,j)} (x-i)^k (y-j)^\ell \quad (6.7)$$

$$\begin{array}{l} i \leq x \leq i+1 \\ j \leq y \leq j+1 \end{array}$$

The  $a_{k,\ell}^{(i,j)}$  are computed with the aid of (A.13) to (A.15). One can pre-compute the  $a_{k,\ell}^{(i,j)}$  for as many meshes as storage allows. They can always be computed from the  $h_{ij}$ ,  $m_{ij}$ ,  $n_{ij}$ ,  $p_{i,j}$  which do not require that much storage (one-fourth).

Turning now toward the  $G_I$ -integral take a plane approximation to formulas (1.3) and (3.18):

$$G_I(x,y) = \frac{1}{2\pi} \int_{(x-\bar{x})^2 + (y-\bar{y})^2 \leq R^2} \frac{[h(\bar{x},\bar{y}) - h(x,y)] \Delta g(\bar{x},\bar{y})}{[(x-\bar{x})^2 + (y-\bar{y})^2]^{3/2}} d\bar{x} d\bar{y} \quad (6.8)$$

$$G_{IC}(x,y) = \frac{1}{2\pi} \int_{(x-\bar{x})^2 + (y-\bar{y})^2 \leq \rho^2} \left\{ \frac{1}{\sqrt{(x-\bar{x})^2 + (y-\bar{y})^2}} - \frac{1}{\rho} \right\} \\ \cdot \text{lap}_{\bar{x},\bar{y}} [h(\bar{x},\bar{y}) - h(x,y) \Delta g(\bar{x},\bar{y})] d\bar{x} d\bar{y} - \\ - \frac{1}{2\pi} \frac{1}{\rho^2} \oint_{\partial C} [h(\bar{x},\bar{y}) - h(x,y)] \Delta g(\bar{x},\bar{y}) d\partial C(\bar{x},\bar{y}) \quad (6.9)$$

Converting everything to polar coordinates centered at  $x,y$  gives:

$$G_1(x, y) = \frac{1}{2\pi} \int_{r=0}^R \int_{\alpha=0}^{2\pi} [h(x + r \cos \alpha, y + r \sin \alpha) - h(x, y)] \cdot \Delta g(x + r \cos \alpha, y + r \sin \alpha) \frac{dr}{r^2} d\alpha \quad (6.10)$$

$$G_{1C}(x, y) = \frac{1}{2\pi} \int_{r=0}^{\rho} \int_{\alpha=0}^{2\pi} \left[1 - \frac{r}{\rho}\right] \text{lap}_{\bar{x}, \bar{y}} [(h(\bar{x}, \bar{y}) - h(x, y)) \Delta g(\bar{x}, \bar{y})]_{\substack{\bar{x}=x+r \cos \alpha \\ \bar{y}=y+r \sin \alpha}} \cdot dr d\alpha$$

$$- \frac{1}{2\pi} \frac{1}{\rho} \int_{\alpha=0}^{2\pi} [h(x + r \cos \alpha, y + r \sin \alpha) - h(x, y)] \Delta g(x + r \cos \alpha, y + r \sin \alpha) d\alpha \quad (6.11)$$

I have used the following procedure for the evaluation of the integrals. In (6.10) the integration with respect to  $\alpha$  has been carried out first using the trapezoidal rule. (The trapezoidal rule is advantageous for periodic functions.) Integration takes place for a certain set of discrete  $r$ -values. Afterwards the trapezoidal rule has been applied to integrate over  $r$ . Values for  $h$  and  $\Delta g$  have been computed from spline formulas throughout. (This has only been done for checking purposes. Later on we will use a less complicated procedure for the more distant areas.) The first integral in (6.11) has been treated similarly. The expression involving the Laplacian has been expanded as:

$$\begin{aligned} \text{lap}_{\bar{x}, \bar{y}} [(h(\bar{x}, \bar{y}) - h(x, y)) \Delta g(\bar{x}, \bar{y})] &= [h(\bar{x}, \bar{y}) - h(x, y)] \text{lap} \Delta g(\bar{x}, \bar{y}) \\ &+ \Delta g(\bar{x}, \bar{y}) \text{lap} h(\bar{x}, \bar{y}) + 2 (\text{grad} h(\bar{x}, \bar{y}), \text{grad} \Delta g(\bar{x}, \bar{y})) \end{aligned} \quad (6.12)$$

The various derivatives have been computed from the spline interpolation formulas. (Note that the bicubic spline has continuous second derivatives.) The second integral in (6.11) has also been approximated with the aid of the trapezoidal rule.

For  $N=21$  we took  $x=11$ ,  $y=11$ ,  $\rho=.8$ ,  $R=3.9$ . The result was

$$G_1(11,11) = - \frac{1}{2\pi} 0.1525$$

For checking purposes we also took  $\rho=5$ . The result was then

$$G_1(11,11) = - \frac{1}{2\pi} 0.1510$$

The various stepsizes for the numerical integration were

$$\text{for } r \leq \rho : \Delta r = \frac{\rho}{8}, \quad \Delta \alpha = \frac{2\pi}{4 r / \Delta r}$$

$$\text{for } r \geq \rho : \Delta r_k = \text{const. } 1.3^{k-1}, k = 1, \dots, 8, \text{ const} = \frac{R - \rho}{\frac{1.3^8 - 1}{0.3}}$$

$$\Delta \alpha = 2\pi/32$$

## APPENDIX A: Some Aspects of Spline Interpolation

It is not our intention to give a systematic account of spline interpolation. Books like Ahlberg et al. (1967) may be consulted for a thorough discussion. The idea behind spline interpolation and some of its advantages shall be outlined. Formulas and methods used in the main portion of this report shall be documented.

### A.1 The Cubic Spline

Let us start with a sequence of  $n$  equidistant abscissa's  $x_i$ ,  $i=1, \dots, n$  on the  $x$ -axis of a two-dimensional coordinate system. Let the distance  $x_{i+1} - x_i$  be denoted by  $h$ . Let function values  $y_i$  be prescribed for each  $x_i$ . The old problem of interpolating these discrete function values can be solved in many ways. One way is interpolation by fitting a polynomial of degree  $n-1$  through the  $n$  points. This can be done e.g. using a polynomial  $p_i(x)$  of degree  $n-1$  having the properties  $p_i(x_j) = 1$  for  $i=j$  and  $p_i(x_j) = 0$  otherwise. Analytical expression for these polynomials are found under "Lagrange interpolation" in any relevant textbook. The interpolating polynomial is then given by

$$p(x) = \sum_{i=1}^n y_i p_i(x) \quad (A.1)$$

An advantage is the analyticity of the resulting expression which is infinitely differentiable. A disadvantage is the following. A change (an error) in one of the  $y$ 's causes considerable changes (errors) over most of the whole range. This is immediately seen from equation (A.1). A change  $\Delta y_{i_0}$  in  $y_{i_0}$  causes a change in  $p(x)$  which is given by

$$\Delta p(x) = \Delta y_{i_0} p_{i_0}(x) \quad (A.2)$$

This may not be too bad as far as only function values are concerned. The changes may be much larger for the derivatives. An extremely opposite approach would be piecewise linear interpolation. The interpolated function is then between  $x_i$  and  $x_{i+1}$  given by the straight line.

$$l(x) = y_i \frac{x - x_{i+1}}{x_i - x_{i+1}} + y_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \quad (\text{A.3})$$

A change in  $y_i$  causes now only changes in  $x_{i-1} \leq x \leq x_{i+1}$  and no further reverberations. A disadvantage is evident from the fact that the function is not continuously differentiable over the entire range.

Spline interpolation tries to compromise between these two extremes. The interpolating function is assumed to be a polynomial of odd degree  $k$  in each of the intervals  $x_i \leq x \leq x_{i+1}$ . However, these polynomials vary from interval to interval. The continuity of the function as well as of its first  $k-1$  derivatives is required throughout, i.e. also for the abscisses  $x_i$ ,  $i=1, \dots, n$ . Certain boundary conditions for  $x_1$ ,  $x_n$  may be prescribed in order to make the problem unique. This interpolating function  $s(x)$  is then  $(k-1)$  times continuously differentiable and, as we shall see later, a change in one of the  $y_i$  causes changes in  $s(x)$  which are heavily dampened for larger distances  $x - x_i$ .

We discuss here only the case  $k=3$  which is known under the label "cubic spline". (See Ahlberg et al section 2.1). We have to choose some parameters which characterize the polynomials in each interval. Let these parameters be  $s(x_i) = y_i$  and  $s'(x_i) = m_i$ , say. This is a meaningful choice since  $s(x)$  and  $s'(x_i)$  are the same for the two intervals  $x_{i-1} \leq x \leq x_i$  and  $x_i \leq x \leq x_{i+1}$ . Besides that, the four quantities  $y_i$ ,  $m_i$ ,  $y_{i+1}$ ,  $m_{i+1}$  determine the four parameters of a cubic polynomial in  $x_i \leq x \leq x_{i+1}$  completely. The continuity requirement for the second derivative at points  $x_2, x_3, \dots, x_{n-1}$  leads to the following set of equations.

$$\frac{1}{2} m_{i-1} + 2 m_i + \frac{1}{2} m_{i+1} = \frac{3}{2h} (y_{i+1} - y_{i-1}) \quad i=2, \dots, n-1 \quad (\text{A.4})$$

A detailed derivation of this equation is given in the above stated reference. However, the principle of how these equations are derived should be clear. The two polynomials in  $x_{i-1} \leq x \leq x_i$  and  $x_i \leq x \leq x_{i+1}$  are linearly expressed in terms of  $y_{i-1}$ ,  $y_i$ ,  $y_{i+1}$ ;  $m_{i-1}$ ,  $m_i$ ,  $m_{i+1}$  and then the second derivatives at  $x = x_i$  are equated.

Since there are in (A.4)  $n-2$  independent equations for  $n$  unknowns, two more independent conditions are needed. They are chosen involving only the "boundary values"  $m_1$ ,  $m_2$  and  $m_{n-1}$ ,  $m_n$ . However we shall not elaborate on this.

Having chosen these additional conditions in some way the quantities  $m_i$  can be solved for which leads to expressions like

$$m_i = \frac{3}{h} \sum_{j=1}^n \alpha_{ij}^{(n)} y_j \quad (\text{A.5})$$

It is now important that, as it is rigorously proved in Ahlberg et al (1967), section 2.4, for  $n \rightarrow \infty$  and  $i \rightarrow \infty$ ,  $n-i \rightarrow \infty$ , (i.e. for a large number of points and  $i$  far enough from the boundary) the coefficients  $\alpha_{ij}^{(n)}$  tend toward a limit  $\alpha_{j-i}$  which depends only on  $j-i$  and equals

$$\alpha_{j-i} = \begin{cases} -\sigma^{j-i} & \text{for } j-i > 0 \\ 0 & \text{for } j-i = 0 \\ +\sigma^{i-j} & \text{for } j-i < 0 \end{cases} \quad (\text{A.6})$$

where

$$\sigma = -(2 - \sqrt{3}) \approx -0.268 \quad (\text{A.7})$$

Since  $\sigma < 1$  we see that only values  $y_j$  at  $x_j$  in the neighborhood of  $x_i$  contribute significantly toward  $m_i$ ,  $m_{i+1}$  and therefore to the polynomial the interval  $x_i \leq x \leq x_{i+1}$ .

(A.5), (A.6), (A.7) may be comprised to

$$m_i = -\frac{3}{h} \sum_{j=1}^{\beta} \sigma^j (y_{i+j} - y_{i-j}) \quad (\text{A.8})$$

The summation limit  $\beta$  is chosen in a way that  $\sigma^\beta$  is sufficiently small. In the practical applications of cubic splines within this report we found that in view of the limited accuracy of the data (i.e. heights, gravity anomalies) a value of  $\beta = 6$  was sufficient.

## A.2 The Bi-Cubic Spline

For a function depending on two variables a generalization is necessary. A useful approach is to assume that function values  $z_{ij}$  are prescribed for each point  $(x_i, y_j)$  of a rectangular grid. For simplicity we assume in the discussion here that the spacings  $x_{i+1} - x_i$  and  $y_{j+1} - y_j$  are equal and equal to  $h$ .

One can proceed in the following way. Cubic splines  $s_j(x)$  are computed interpolating the given  $z_{ij}$  for fixed  $j$ . Then for a fixed  $x$  a cubic spline  $s(x, y)$  viewed as a function of  $y$  is computed interpolating the  $s_j(x)$  values. In that way for each  $x, y$  an interpolated value, namely,  $s(x, y)$  is obtained. In this procedure the role of  $x$  and  $y$  (or  $i$  and  $j$ ) may be interchanged and, surprisingly, the result is the same, at least for a certain class of boundary conditions. The resulting function  $s(x, y)$  is a bi-cubic polynomial in each quadrangle  $x_i \leq x \leq x_{i+1}$ ,  $y_i \leq y \leq y_{i+1}$ , i.e.:

$$s(x, y) = \sum_{k, \ell=0}^3 a_{k, \ell}^{(i, j)} (x - x_i)^k (y - y_j)^\ell \quad \begin{matrix} x_i \leq x \leq x_{i+1} \\ y_j \leq y \leq y_{j+1} \end{matrix} \quad (\text{A.9})$$

$s(x, y)$  is together with its first and second derivatives continuous over the entire range. One way to compute the  $a_{k, \ell}^{(i, j)}$  is the following. Perform an ordinary spline interpolation along the lines  $y = y_j$ ,  $x = x_i$ . Obtain from that according to section A.2 the values

$$m_{ij} = \left. \frac{\partial s(x, y_j)}{\partial x} \right|_{x=x_i} \quad (\text{A.10})$$

$$n_{ij} = \left. \frac{\partial s(x_i, y)}{\partial y} \right|_{y=y_j} \quad (\text{A.11})$$

Perform an ordinary spline interpolation of  $m_{ij}$  for fixed  $i$  and obtain from that

$$p_{ij} = \left. \frac{\partial^2 s(x, y)}{\partial x \partial y} \right|_{x=x_i, y=y_j} \quad (\text{A.12})$$

The same result would be obtained if a spline interpolation on  $n_{ij}$  for fixed  $j$  was performed. Now compute the matrix  $a^{ij}$  from the formulas

$$a^{(i,j)} = A^T(h) K^{(i,j)} A(h) \quad (A.13)$$

with

$$A(h) = \begin{bmatrix} 1 & 0 & -3/h^2 & 2/h^3 \\ 0 & 1 & -2/h & 1/h^2 \\ 0 & 0 & 3/h^2 & -2/h^3 \\ 0 & 0 & -1/h & 1/h^2 \end{bmatrix} \quad (A.14)$$

and

$$K^{(i,j)} = \begin{bmatrix} z_{i,j} & n_{i,j} & z_{i,j+1} & n_{i,j+1} \\ m_{i,j} & p_{i,j} & m_{i,j+1} & p_{i,j+1} \\ z_{i+1,j} & n_{i+1,j} & z_{i+1,j+1} & n_{i+1,j+1} \\ m_{i+1,j} & p_{i+1,j} & m_{i+1,j+1} & p_{i+1,j+1} \end{bmatrix} \quad (A.15)$$

These formulas are found in DeBoor (1962) and have also been used in Davis and Kontis (1970).

It is clear that boundary conditions influence the values  $m_{ij}$ ,  $n_{ij}$ ,  $p_{ij}$  and therefore  $a_{kl}^{(ij)}$ . However, this influence is nearly zero if from now on we assume that the grid is very large and the quadrangle is at a sufficient distance from the boundary. In that case the quantities  $m_{ij}$ ,  $n_{ij}$ ,  $p_{ij}$  can be in agreement with section (A.1) and can be computed from

$$m_{ij} = - \frac{3}{h} \sum_{k=1}^{\beta} \sigma^k (z_{i+k,j} - z_{i-k,j}) \quad (A.16)$$

$$n_{ij} = - \frac{3}{h} \sum_{\ell=1}^{\beta} \sigma^{\ell} (z_{i,j+\ell} - z_{i,j-\ell}) \quad (A.17)$$

$$p_{ij} = - \frac{3}{h} \sum_{\ell=1}^{\beta} \sigma^{\ell} (m_{i,j+\ell} - m_{i,j-\ell}) \quad (A.18)$$

or alternatively

$$p_{ij} = - \frac{3}{h} \sum_{k=1}^{\beta} \sigma^k (n_{i+k,j} - n_{i-k,j}) \quad (A.19)$$

This, together with (A.13) to (A.15) was the set of formulas used in this report. Generalizations to grids with unequal spacings are possible. However, one can use the equal spacing formulas also for an originally unequally spaced grid, or even for a curvi-linear grid applying to some curved surface. The derived interpolation polynomial (A.9) has only to be interpreted properly. We illustrate this in the case of the unit sphere with a  $\theta, \lambda$  coordinate system. Let the grid be formed of  $\theta, \lambda$  lines at distances of, let's say,  $1^\circ$ . This grid is not plane nor is it equally spaced. Let  $z_{ij}$  nevertheless refer to the grid points and derive the polynomial (A.9) by using the set of formulas (A.10) - (A.15) with  $h$  equal to  $\pi/180$  (arc length of one degree at the equator). Let  $s(\theta, \lambda)$  be this polynomial where  $\theta, \lambda$  (arc lengths) replace formally  $x, y$ :

$$s(\theta, \lambda) = \sum_{k=0}^3 \sum_{\ell=0}^3 a_{k,\ell}^{(i,j)} (\theta - \theta_i)^k (\lambda - \lambda_j)^\ell \quad \begin{matrix} \theta_i \leq \theta \leq \theta_{i+1} \\ \lambda_j \leq \lambda \leq \lambda_{j+1} \end{matrix} \quad (A.20)$$

There is nothing against evaluating  $s(\theta, \lambda)$  for any  $\theta, \lambda$  value not coinciding with a grid point from this formula. Also derivatives with respect to  $\theta, \lambda$  can be evaluated by formally differentiating (A.20). These derivatives can be used in various vector analytical expressions like e.g.

$$\text{Lap } s(\theta, \lambda) = \frac{\partial^2 s(\theta, \lambda)}{\partial \theta^2} + \frac{\partial s(\theta, \lambda)}{\partial \theta} \cot \theta + \frac{\partial^2 s(\theta, \lambda)}{\partial \lambda^2} \frac{1}{\sin^2 \theta} \dots \quad (A.21)$$

## APPENDIX B: Isotropic Integral-Operators With Truncated Kernel Functions

### B.1 Introduction

Let  $\Gamma$  be the unit sphere and let the unit vectors  $\xi, \eta$  be points on  $\Gamma$ . A function  $K(\xi, \eta)$  is called a kernel function since it may occur in an integral transformation

$$g(\xi) = \int_{\Gamma} K(\xi, \eta) f(\eta) d\Gamma(\eta) \quad (B.1)$$

If  $K(\xi, \eta)$  depends only on the distance between  $\xi, \eta$  or, equivalently, only on the inner product

$$\cos \psi = \xi \cdot \eta \quad (B.2)$$

then  $K(\xi, \eta)$  will be written as  $K(\xi \cdot \eta)$  or  $K(\cos \psi)$  and will be called an isotropic kernel. It is known that the spherical harmonics  $S_{nm}(\xi)$  are the eigen functions of such integral transformations:

$$\int_{\Gamma} K(\xi \cdot \eta) S_{nm}(\eta) d\Gamma(\eta) = \lambda_n S_{nm}(\xi) \quad (B.3)$$

with

$$\lambda_n = 2\pi \int_{-1}^{+1} K(t) P_n(t) dt \quad (B.4)$$

$P_n(t)$  is the Legendre polynomial. (B.3) together with (B.4) is called Funk-Hecke formula in Müller (1966). If the spherical harmonics expansions of  $f(\xi)$  and  $g(\xi)$  are

$$f(\xi) = \sum_{n,m} f_{nm} S_{nm}(\xi), \quad g(\xi) = \sum_{n,m} g_{nm} S_{nm}(\xi)$$

Then (B.1) can be replaced by

$$g_{nm} = \lambda_n f_{nm} \quad (B.4a)$$

This follows immediately from (B.1), (B.3). Truncation of an integral transformation (B.3) means that in the formula

$$g(\xi) = \int_{\Gamma} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.5)$$

the integration is carried out only over some (usually circular) cap around  $\xi$ . Let  $C = C(\xi, \psi_0)$  denote a circular cap centered at  $\xi$  and having spherical radius  $\psi_0$ . Then instead of (B.5) we have

$$\tilde{g}(\xi) = \int_C K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.6)$$

Introducing the truncated kernel

$$\tilde{K}(\xi \cdot \eta) = \begin{cases} K(\xi \cdot \eta) & \text{for } \xi \cdot \eta \geq \cos \psi_0 \\ 0 & \text{for } \xi \cdot \eta < \cos \psi_0 \end{cases} \quad (B.7)$$

the equation (B.7) can also be written as

$$\tilde{g}(\xi) = \int_{\Gamma} \tilde{K}(\xi \cdot \eta) f(\eta) d\Gamma(\eta)$$

This shows that  $\tilde{K}(\xi \cdot \eta)$  is also an isotropic operator, hence according to the Funk-Hecke formula

$$\int_{\Gamma} \tilde{K}(\xi \cdot \eta) S_{nm}(\eta) d\Gamma(\eta) = \tilde{\lambda}_n S_{nm}(\xi) \quad (B.8)$$

$$\tilde{\lambda}_n = 2\pi \int_{-1}^{+1} \tilde{K}(t) P_n(t) dt \quad (B.9)$$

The error function

$$\Delta g(\xi) = g(\xi) - \tilde{g}(\xi) \quad (B.10)$$

is then given by

$$\Delta g(\xi) = \int_{\Gamma-C} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) = \int_{\Gamma} \Delta K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.11)$$

with

$$\Delta K(\xi \cdot \eta) = K(\xi \cdot \eta) - \tilde{K}(\xi \cdot \eta) \quad (B.12)$$

Its eigen values are

$$\Delta \lambda_n = \lambda_n - \tilde{\lambda}_n$$

with

$$\Delta \lambda_n = 2\pi \int_{-1}^{+1} \Delta K(t) P_n(t) dt = 2\pi \int_{-1}^{\cos \psi_0} K(t) P_n(t) dt \quad (B.13)$$

In case of Stokes' operator, i.e.

$$K(\xi \cdot \eta) = \frac{1}{4\pi} \text{St}(\xi \cdot \eta)$$

with Stokes' function defined by

$$\text{St}(\xi \cdot \eta) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\xi \cdot \eta) \quad (B.14)$$

the  $\Delta \lambda_n$  are given by  $\frac{1}{2} Q_n$  where  $Q_n$  are the Molodensky coefficients

$$Q_n = \int_{-1}^{\cos \psi_0} \text{St}(t) P_n(t) dt \quad (B.15)$$

The usual method to evaluate these coefficients is based on Molodensky et al (1962) and is designed to yield  $Q_n = Q_n(\psi_0)$  for fixed  $n$  and all  $\psi_0$ . In section B3-3 we shall give a recursion formula which will yield  $Q_n(\psi_0)$  for fixed  $\psi_0$  and theoretically all  $n$ .

We shall also deal with the following question: Is there a way to decrease the truncation error by truncating not abruptly at  $\psi_0$  but continuously over an integral  $\psi_1 \leq \psi \leq \psi_0$ . In that case we would put

$$\begin{aligned} \tilde{K}(\cos \psi) &= K(\cos \psi) & 0 \leq \psi \leq \psi_1 \\ K(\cos \psi) &= 0 & \psi_0 \leq \psi \leq \pi \end{aligned}$$

and we would choose  $\tilde{K}(\cos \psi)$  somehow in the interval  $\psi_1 \leq \psi \leq \psi_0$  so that  $\tilde{K}(\cos \psi)$  or, equivalently,  $\Delta K(\cos \psi)$  has some desirable properties like continuity or continuous differentiability.

It is known that generally the Legendre coefficients decrease more rapidly

to zero as  $n$  goes to infinity if the relevant function is smooth. Therefore  $\Delta\lambda_n$ , the eigen values of the error kernel

$$\Delta K(\cos \psi) = K(\cos \psi) - \tilde{K}(\cos \psi) \quad (\text{B.16})$$

can be expected to taper off more quickly. This is for example seen from

$$\Delta\lambda_n = 2\pi \int_{-1}^{+1} \Delta K(t) P_n(t) dt = \int_{\Gamma} \Delta K(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta)$$

by Green's theorem (2.5) for  $n \geq 1$ :

$$\begin{aligned} &= \int_{\Gamma} \text{Lap}_{\eta} \Delta K(\xi \cdot \eta) \text{Lap}_{\eta}^{-1} P_n(\xi \cdot \eta) d\Gamma(\eta) = \\ &= \frac{1}{n(n+1)} \int_{\Gamma} \text{Lap}_{\eta} \Delta K(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \end{aligned}$$

If  $\Delta K$  is two times differentiable so that the surface-Laplacean  $\text{Lap} \Delta K$  exists and is squared integrable, then it follows from the Schwarz inequality that

$$\Delta\lambda_n \leq \frac{1}{n(n+1)} \sqrt{\int_{\Gamma} (\text{Lap}_{\eta} \Delta K(\xi \cdot \eta))^2 d\Gamma(\eta)} \sqrt{\int_{\Gamma} (P_n(\xi \cdot \eta))^2 d\Gamma(\eta)}$$

Thus, since the integral involving  $P_n$  is  $O(\frac{1}{n})$ :

$$\Delta\lambda_n = O\left(\frac{1}{n^2\sqrt{n}}\right) \quad (\text{B.17})$$

In section (B.4) we shall deal with this question more systematically. One of the necessary preparations will be to compute the surface Laplacean for certain frequently used kernel functions  $K(\xi \cdot \eta)$ .

## B.2 The Surface Laplacean of Some Kernel Functions

If  $K(\xi \cdot \eta)$  is an isotropic kernel function, we are interested in the quantity

$$\text{Lap}_\eta K(\xi \cdot \eta)$$

which denotes the surface Laplacean with respect to  $\eta$ . (Because of the complete symmetry of the problem the Laplacean with respect to  $\xi$  would give the same result.)

If we put

$$\eta = (\sin \theta \cos \lambda, \sin \theta \sin \lambda, \cos \theta)^T$$

and if  $F(\theta, \lambda)$  denotes a function on  $\Gamma$ , depending now on polar distance angle  $\theta$  and longitude  $\lambda$ , then the Laplacean may be computed from

$$\text{Lap } F(\theta, \lambda) = F_{\theta\theta} + F_\theta \cot \theta + F_{\lambda\lambda} \frac{1}{\sin^2 \theta} \quad (\text{B.18})$$

In our application we assume the north pole of the  $\theta, \lambda$  system in  $\xi$ .  $K(\xi \cdot \eta)$  may then be written as  $K(\cos \theta)$  and is independent of  $\lambda$ . Thus

$$\text{Lap}_\eta K(\xi \cdot \eta) = \text{Lap } K(\cos \theta) = K_{\theta\theta} + K_\theta \cot \theta \quad (\text{B.19})$$

B.2-1 The Kernel  $1/\ell(\xi \cdot \eta)$ . Let  $\ell(\xi \cdot \eta)$  denote the distance between  $\xi, \eta$ , i.e. with  $\xi \cdot \eta = \cos \varphi$

$$\ell(\xi \cdot \eta) = \xi - \eta = 2 \sin \frac{\psi}{2} \quad (\text{B.20})$$

Straight forward differentiation according to (B.19) and (B.20) with  $\psi = \theta$  yields the result

$$\text{Lap}_\eta \frac{1}{\ell(\xi \cdot \eta)} = \frac{1}{\ell^3(\xi \cdot \eta)} + \frac{1}{4 \ell(\xi \cdot \eta)} \quad (\text{B.21})$$

B.2-2 The Kernel  $\ell(\xi \cdot \eta)$ . In an analogous manner we obtain

$$\text{Lap}_\eta \ell(\xi \cdot \eta) = \frac{1}{\ell(\xi \cdot \eta)} - \frac{3 \ell(\xi \cdot \eta)}{4} \quad (\text{B.22})$$

B.2-3 The Kernel  $2\ln(2/\ell(\xi \cdot \eta))$ . The result for this kernel is simply

$$\text{Lap}_\eta \left( 2\ln \frac{2}{\ell(\xi \cdot \eta)} \right) = 1 \quad (\text{B.23})$$

B.2-4 Stokes' Function. The Laplacean of Stokes' function

$$\begin{aligned} \text{St}(\cos \psi) &= \frac{1}{\sin \frac{\psi}{2}} - 6 \sin \frac{\psi}{2} + 1 - 5 \cos \psi \\ &\quad - 3 \cos \psi \ln \left( \sin \frac{\psi}{2} + \sin^2 \left( \frac{\psi}{2} \right) \right) \end{aligned} \quad (\text{B.24})$$

could also be obtained by direct differentiation. It is more elegant perhaps to use an alternative method which utilizes the spherical harmonics expansion (B.14). Since

$$\text{Lap}_\eta P_n(\xi \cdot \eta) = -n(n+1) P_n(\xi \cdot \eta) \quad (\text{B.25})$$

it is tempting to perform the Laplacean termwise in (B.14). This is, however, not feasible since the resulting series does not even converge. We proceed therefore in the following way. Call temporarily

$$f(\xi \cdot \eta) = \frac{1}{\ell(\xi \cdot \eta)} - P_0(\xi \cdot \eta) - P_1(\xi \cdot \eta) = \sum_{n=2}^{\infty} P_n(\xi \cdot \eta) \quad (\text{B.26})$$

and (Meissl (1971b, equ. (6.50)),

$$g(\xi \cdot \eta) = 2\ln \frac{2}{\ell(\xi \cdot \eta)} - \frac{3}{2} P_1(\xi \cdot \eta) = \sum_{n=2}^{\infty} \frac{2n+1}{n(n+1)} P_n(\xi \cdot \eta) \quad (\text{B.27})$$

Now form the function

$$\begin{aligned} \text{St}(\xi \cdot \eta) - 2f(\xi \cdot \eta) - \frac{3}{2} g(\xi \cdot \eta) &= \\ \frac{9}{4} \sum_{n=2}^{\infty} \frac{2n+1}{(n+\frac{1}{2})n(n+1)} P_n(\xi \cdot \eta) &+ 3 \sum_{n=2}^{\infty} \frac{2n+1}{(n-1)(n+\frac{1}{2})n(n+1)} P_n(\xi \cdot \eta) \\ &\dots \quad (\text{B.28}) \end{aligned}$$

which may be verified taking into account the spherical harmonics expansions of the involved functions. The idea behind (B.28) is that the Laplacean can be applied term-wise on the right-hand side while the left-hand side causes no problems in view of (B.21) and (B.23). Performing the necessary elementary manipulations one arrives finally at

$$\text{Lap}_\eta \text{St}(\xi \cdot \eta) = \frac{2}{\ell^3(\xi \cdot \eta)} - 2 \text{St}(\xi \cdot \eta) + 2 P_0(\xi \cdot \eta) + 9 P_1(\xi \cdot \eta) \quad (\text{B.29})$$

**Remark:** The procedure can easily be extended to residual Stokes' kernel functions which are obtained after eliminating from  $\text{St}(\xi \cdot \eta)$  spherical harmonics up to and including degree  $N$ . Call

$$\text{St}_N(\xi \cdot \eta) = \text{St}(\xi \cdot \eta) - \sum_{n=2}^N \frac{2n+1}{n-1} P_n(\xi \cdot \eta) = \sum_{n=N+1}^{\infty} \frac{2n+1}{n-1} P_n(\xi \cdot \eta) \quad (\text{B.30})$$

Then (B.29) generalizes to

$$\text{Lap}_\eta \text{St}_N(\xi \cdot \eta) = \frac{2}{\ell^3(\xi \cdot \eta)} - 2 \text{St}_N(\xi \cdot \eta) + \sum_{n=0}^N (2n^2 + 5n + 2) P_n(\xi \cdot \eta) \quad (\text{B.31})$$

### B.3 Truncation at a Certain Angle $\psi_0$ .

In this section we concentrate on the error kernel  $\Delta K(\xi \cdot \eta)$  defined in (B.16) having eigen values  $\Delta \lambda_n$  according to (B.13):

$$\Delta \lambda_n = 2\pi \int_{-1}^{+1} \Delta K(t) P_n(t) dt = 2\pi \int_{-1}^{\cos \psi_0} K(t) P_n(t) dt \quad (\text{B.32})$$

If the truncated kernel is applied to a function  $f(\xi) \sim f_{nm}$  ( $f_{nm}$  - spherical harmonics coefficients) then the resulting error  $\Delta g(\xi) \sim \Delta g_{nm}$  is given by

$$\Delta g(\xi) = \int_{\Gamma} \Delta K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (\text{B.33})$$

or in view of (B.4a) by

$$\Delta g_{nm} = \Delta \lambda_n f_{nm} \quad (\text{B.34})$$

We shall now compute the  $\Delta \lambda_n$ 's for some kernels:

B.3-1. The Kernel  $1/\ell(\xi, \eta)$ : From (B.20) and (B.32) we get with  $\cos(\psi_0) = t_0$  because of  $\sin \frac{\psi}{2} = \sqrt{\frac{1 - \cos \psi}{2}}$

$$\Delta \lambda_n = \sqrt{2} \pi \int_{-1}^{t_0} \frac{1}{\sqrt{1-t}} P_n(t) dt \quad (\text{B.35})$$

Write

$$b_n = \int_{-1}^{t_0} \frac{1}{\sqrt{1-t}} P_n(t) dt \quad (\text{B.36})$$

Then

$$\Delta \lambda_n = \sqrt{2} \pi b_n \quad (\text{B.36a})$$

We derive now a recursion formula for  $b_n$ . First, for  $n = 0, 1$  one computes in an elementary way

$$\begin{aligned} b_0 &= -2\sqrt{1-t_0} + 2\sqrt{2} \\ b_1 &= \frac{2\sqrt{(1-t_0)^3}}{3} - 2\sqrt{1-t_0} + \frac{2\sqrt{2}}{3} \end{aligned} \quad (\text{B.37})$$

For  $n \geq 2$  we replace  $P_n(t)$  by

$$P_n(t) = \frac{2n-1}{n} t P_{n-1}(t) - \frac{n-1}{n} P_{n-2}(t)$$

cf. Abramowitz-Stegun (1964), formula (8.5.3). Inserting into (B.36) yields

$$\begin{aligned}
b_n &= \frac{2n-1}{n} \int_{-1}^{t_0} \frac{t}{\sqrt{1-t}} P_{n-1}(t) dt - \frac{n-1}{n} b_{n-2} \\
&= - \frac{2n-1}{n} \int_{-1}^{t_0} \sqrt{1-t} P_{n-1}(t) dt + \frac{2n-1}{n} b_{n-1} - \frac{n-1}{n} b_{n-2}
\end{aligned}$$

Applying partial integration and using

$$\int_{-1}^{+1} P_n(t) dt = \frac{1}{2n+1} (P_{n+1}(t_0) - P_{n-1}(t_0))$$

(cf. Lense (1954), p. 17) we get

$$\begin{aligned}
b_n &= \frac{P_{n-2}(t_0) - P_n(t_0)}{n} \sqrt{1-t_0} - \frac{1}{2n} (b_n - b_{n-2}) + \frac{2n-1}{n} b_{n-1} \\
&\quad - \frac{n-1}{n} b_{n-2}
\end{aligned}$$

solving for  $b_n$  yields

$$b_n = \frac{P_{n-2}(t) - P_n(t)}{n + \frac{1}{2}} \sqrt{1-t_0} + 2 \frac{n - \frac{1}{2}}{n + \frac{1}{2}} b_{n-1} - \frac{n - \frac{3}{2}}{n + \frac{1}{2}} b_{n-2} \quad (B.38)$$

This is the desired recurrence relation.  $b_n$  could be replaced by  $\Delta\lambda_n$  using (B.36a)

B.3-2 The Kernel  $1/\ell^3(\xi, \eta)$ . For this kernel we have

$$\Delta\lambda_n = \frac{\pi}{\sqrt{2}} \int_{-1}^{t_0} \frac{1}{\sqrt{(1-t)}^3} P_n(t) dt \quad (B.39)$$

Writing

$$\Delta\lambda_n = \frac{\pi}{\sqrt{2}} c_n \quad (B.40)$$

with

$$c_n = \int_{-1}^{t_0} \frac{1}{\sqrt{(1-t)}^3} P_n(t) dt \quad (B.41)$$

One derives in an analogous way the recurrence relations

$$c_0 = \frac{2}{\sqrt{1-t_0}} - \sqrt{2} \quad (B.42)$$

$$c_1 = \frac{2}{\sqrt{1-t_0}} + 2\sqrt{1-t_0} - 3\sqrt{2}$$

$$c_n = \frac{P_{n-2}(t_0) - P_n(t_0)}{(n - \frac{1}{2})\sqrt{1-t_0}} + 2c_{n-1} - c_{n-2} \quad (B.43)$$

B.3-3 Stokes' Kernel. We take a slightly different approach and write the equation (B.32) in the form

$$\Delta \lambda_n = \int_{\Gamma-C} St(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta)$$

$\Gamma-C$  is the complementary area after removal of a cap with spherical radius  $\psi_0$  from the unit sphere  $\Gamma$ . Next we apply Green's second formula and obtain for  $n \geq 1$

$$\begin{aligned} \Delta \lambda_n &= \int_{\Gamma-C} \text{Lap}_\eta St(\xi \cdot \eta) \text{Lap}_\eta^{-1} P_n(\xi \cdot \eta) d\Gamma(\eta) \\ &+ \oint_\beta St(\xi \cdot \eta) (\text{Grad}_\eta (\text{Lap}_\eta^{-1} P_n(\xi \cdot \eta)), \nu(\eta)) d\beta(\eta) \\ &- \oint_\beta \text{Lap}_\eta^{-1} P_n(\xi \cdot \eta) (\text{Grad}_\eta St(\xi \cdot \eta), \nu(\eta)) d\beta(\eta) \end{aligned}$$

$\beta$  is the boundary of  $\Gamma-C$  and  $\nu$  is the unit normal to the boundary, tangential to  $\Gamma$  and outward of  $\Gamma-C$  (inward  $C$ ). Choosing a  $\theta, \lambda$  coordinate system with pole in  $\xi$ , substituting in the line integrals  $t = \cos \theta$ , noting  $\text{Lap}_\eta^{-1} P_n(\xi \cdot \eta) = -(1/(n(n+1))) P_n(\xi \cdot \eta)$ , introducing  $t_0 = \cos \varphi_0$ , and verifying

$$(\text{Grad}_\eta F(\xi \cdot \eta), \nu(\eta)) = - \frac{dF(\cos \theta)}{d\theta} = + \frac{dF(t)}{dt} \sqrt{1-t^2}$$

leads to

$$\begin{aligned}
 \Delta \lambda_n = & - \frac{1}{n(n+1)} \int_{\Gamma-C} \text{Lap}_\eta \text{St}(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \\
 & - \frac{2\pi}{n(n+1)} \text{St}(t_0) \left. \frac{dP_n(t)}{dt} \right|_{t=t_0} (1-t_0^2) \\
 & + \frac{2\pi}{n(n+1)} \left. \frac{d\text{St}(\xi \cdot \eta)}{dt} \right|_{t=t_0} P_n(t_0) (1-t_0^2) \quad (\text{B.44a})
 \end{aligned}$$

Denoting the derivatives with respect to  $t$  by dashes using (B.29) for  $\text{Lap St}$  gives

$$\begin{aligned}
 \Delta \lambda_n = & - \frac{2}{n(n+1)} \int_{\Gamma-C} \frac{1}{\ell^3(\xi \cdot \eta)} P_n(\xi \cdot \eta) d\Gamma(\eta) + \frac{2}{n(n+1)} \int_{\Gamma-C} \text{St}(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \\
 & - \frac{2}{n(n+1)} \int_{\Gamma-C} P_n(\xi \cdot \eta) d\Gamma(\eta) - \frac{9}{n(n+1)} \int_{\Gamma-C} P_1(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \\
 & - \frac{2\pi}{n(n+1)} \text{St}(t_0) P_n'(t_0) (1-t_0^2) + \frac{2\pi}{n(n+1)} \text{St}'(t_0) P_n(t_0) (1-t_0^2) \quad (\text{B.44})
 \end{aligned}$$

Using the results of section B.3-2 and equation (B.44) leads to

$$\begin{aligned}
 \Delta \lambda_n = & - \frac{\sqrt{2}\pi}{n(n+1)} c_n + \frac{2}{n(n+1)} \Delta \lambda_n - \frac{4\pi}{n(n+1)} \int_{t=1}^{t_0} P_n(t) dt \\
 & - \frac{18\pi}{n(n+1)} \int_{t=-1}^{t_0} t P_n(t) dt - \frac{2\pi}{n(n+1)} \text{St}(t_0) P_n'(t_0) (1-t_0^2) \\
 & - \frac{2\pi}{n(n+1)} \text{St}'(t_0) P_n(t_0) (1-t_0^2)
 \end{aligned}$$

Replacing  $\Delta \lambda_n$  by the Molodensky coefficients  $Q_n$  according to

$$\Delta \lambda_n = 2\pi Q_n \quad (\text{B.45})$$

and solving for  $Q_n$  gives finally

$$Q_n = - \frac{1}{\sqrt{2}} \frac{1}{n(n+1)-2} c_n - \frac{1}{n(n+1)-2} \sum_{m=0}^1 (2m^2 + 5m + 2) \int_{-1}^{t_0} P_m(t) P_n(t) dt \quad (B.46)$$

$$- \frac{1}{n(n+1)-2} St(t_0) P_n'(t_0) (1-t_0^2) + \frac{1}{n(n+1)-2} St'(t_0) P_n(t_0) (1-t_0^2)$$

This is the desired formula. (It holds also for  $St_N(\xi, \eta)$  instead of  $St(\xi, \eta)$  if the summation over  $m$  is extended to  $N$ .) See (B.30), (B.31). Formula (B.46) is recursive insofar as the  $c_n$  are the result of the recursion (B.42) - (B.43). The two integrals involving  $P_n(t)$  may be computed from the general formula

$$\int_{-1}^{t_0} P_m(t) P_n(t) dt = \frac{P_m'(t_0) P_n(t_0) - P_m(t_0) P_n'(t_0)}{n(n+1) - m(m+1)} \quad \text{for } n \neq m \quad (B.47)$$

This formula follows from

$$\int_{-1}^{+1} P_m(t) P_n(t) dt = \frac{1}{2\pi} \int_{\Gamma} P_m(\xi, \eta) P_n(\xi, \eta) d\Gamma(\eta)$$

after application of Green's second formula. It holds for  $m \neq n$ , i.e. it may be used for  $n > 2$  in (B.46) for  $n = 0, 1$  elementary integrations are performed. The derivative  $P_n'(t_0)$  is obtained from

$$P_n'(t) = - \frac{n}{1-t^2} (t P_n(t) - P_{n-1}(t)) \quad (B.48)$$

(Lense (1954), p. 16) or a similar formula. The derivative of Stokes' function is in a direct way computed from (B.24) observing

$$St'(t) = \frac{d St(\cos \psi)}{d\psi} \cdot \frac{-1}{\sin \psi}, \quad \cos \psi = t \quad (B.49)$$

B.4 Modified Truncation Procedures. Repeating formula (B.44a) for a general kernel  $K(\xi \cdot \eta)$  we get:

$$\begin{aligned} \Delta \lambda_n = & - \frac{1}{n(n+1)} \int_{\Gamma-C} \text{Lap}_{\eta} K(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) - \\ & - \frac{2}{n(n+1)} (1 - t_0^2) K(t_0) P_n'(t_0) + \frac{2}{n(n+1)} (1 - t_0^2) K'(t_0) P_n(t_0) \quad (\text{B.50}) \end{aligned}$$

One sees that the first and also the third term on the right-hand side taper off like an  $O(\frac{1}{n^2})$  as  $n \rightarrow \infty$ , whereas the second term which contains  $P_n'(t_0)$  does not necessarily in view of (B.48). It is, therefore, natural to try to modify the kernel  $K(\cos \varphi)$  in a way that  $K(t_0)$  vanishes. This can be accomplished by simply subtracting

$$k_0 = K(t_0) \quad (\text{B.51})$$

The effect on the original integral

$$g(\xi) = \int_{\Gamma} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (\text{B.52})$$

is independent of  $\xi$  and therefore constant. For

$$\hat{g}(\xi) = \int_{\Gamma} (K(\xi \cdot \eta) - k_0) f(\eta) d\Gamma(\eta) = g(\xi) - \gamma_0 \quad (\text{B.53})$$

with

$$\gamma_0 = k_0 \int_{\Gamma} f(\eta) d\Gamma(\eta) = 4\pi k_0 \bar{f} \quad (\text{B.54})$$

$\bar{f}$  denotes the mean value of  $f(\eta)$  over  $\Gamma$ .

In most applications a constant error in  $g(\xi)$  does not matter at all. In Stokes' formula for example the harmonic of degree zero is eliminated. The scale of the computed geoid has to be obtained from other sources. Therefore a constant error is without concern. It is, however, important that the truncation angle  $\psi_0$  is the same everywhere.

If  $K(t)$  happens to have a zero at  $t = t_0$ , then no subtraction is necessary. DeWitte (1966) found small truncation errors at the zeros of Stokes' function. The theoretical reason for this is clear by now. It is, however, also clear that the error at any other truncation angle is to a large extent a constant over all of  $\Gamma$  if  $k_0$  is subtracted prior to truncation.

One can, of course, also try to modify the kernel  $K(\cos \psi)$  in a way that also the derivative at the angle of truncation vanishes. Let this angle be again denoted by  $\psi_0$ . Then  $K(\cos \psi)$  will be modified in  $\psi_1 \leq \psi \leq \psi_0$  so that

$$\tilde{K}(\cos \psi) = K(\cos \psi), \quad \psi \leq \psi_1$$

$$\tilde{K}(\cos \psi) = 0, \quad \psi \geq \psi_0$$

In  $\psi_1 \leq \psi \leq \psi_0$   $\tilde{K}(\cos \psi)$  will be chosen in a way that  $\tilde{K}(\cos \psi)$  is twice differentiable in  $0 \leq \varphi \leq \pi$ . The error kernel is then

$$\Delta K(\cos \psi) = 0, \quad \psi \leq \psi_1$$

$$\Delta K(\cos \psi) = K(\cos \psi), \quad \psi \geq \psi_0$$

and  $\Delta K(\cos \varphi)$  twice differentiable in  $0 \leq \psi \leq \pi$ .

The question is now, how to choose  $K$  or  $\Delta K$  within the interval. Since the eigen values of the error kernel are now by (B.50):

$$\begin{aligned} \Delta \lambda_n &= - \frac{1}{n(n+1)} \int_{\Gamma} \text{Lap}_{\eta} \Delta K(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \\ &= - \frac{2\pi}{n(n+1)} \int_{-1}^{+1} \text{Lap}_{\eta} \Delta K(t) P_n(t) dt, \end{aligned} \quad (\text{B.55})$$

it will be beneficial if the Laplacian of  $\Delta K$  is as small as possible. Since the Laplacian depends on the second and first derivative it may be good strategy to try to make these derivatives small.

We view the kernel  $\Delta K$  as a function of  $t = \cos \psi$ . With  $t_0 = \cos \varphi_0$ ,  $t_1 = \cos \varphi_1$ , we take a piecewise quadratic polynomial  $\tilde{\Delta K}(t)$  in  $t_0 \leq t \leq t_1$  such that

$$\begin{aligned} \Delta K(t_0) &= K(t_0) \quad , \quad \Delta K'(t_0) = K'(t_0) \\ \Delta K(t_1) &= 0 \quad , \quad \Delta K'(t_1) = 0 \end{aligned} \tag{B.56}$$

and  $\Delta K(t)$  is a polynomial of degree 2 in each of the intervals  $t_0 \leq t \leq t^*$ ,  $t^* \leq t \leq t_1$ .  $t^*$  will be chosen so that  $\Delta K''$  is of equal modulus (but opposite sign) in the two intervals. Of course  $\Delta K(t)$  and  $\Delta K'(t)$  shall be continuous at  $t = t^*$ .

We do not write down the explicit set of formulas which is completely elementary. It may only be mentioned that  $t^*$  may coincide with  $t_1$  which appears desirable since then one of the discontinuities of the second derivative of  $\Delta K(t)$  disappears. This special case may for given  $t_0, t_1$  be obtained by adding a constant to  $\Delta K(t)$ . Adding a constant to  $\Delta K(t)$  amounts to adding one to  $K(t)$  which is of not much concern as we have seen earlier.

More sophisticated kernel - modifications can be conceived. For example the second and third derivatives of the truncated kernel (or, equivalently, of the error kernel) could also be made continuous. This, however, will not be done here.

B.5 Truncation of Vening-Meinesz - Type Integrals . We perform the surface gradient of (B.1), i.e. of

$$g(\xi) = \int_{\Gamma} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.59)$$

and obtain by differentiating under the integral sign

$$w(\xi) = \text{Grad}_{\xi} g(\xi) = \int_{\Gamma} \text{Grad}_{\xi} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.60)$$

If  $K(\cos \psi)$  has a pole of order less than two with respect to  $\psi$  (as for example Stokes' kernel) then (B.60) is still valid. The integral is, however, singular if the order of the pole is greater or equal to 1.

There are two ways of truncating (B.60). Let  $C_{\xi}$  denote a spherical cap around  $\xi$  with half opening angle  $\psi_0$ . One can truncate (B.59):

$$\tilde{g}(\xi) = \int_{C_{\xi}} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) \quad (B.61)$$

and then differentiate

$$\begin{aligned} \tilde{w}(\xi) &= \text{Grad} \tilde{g}(\xi) = \\ &= \int_{C_{\xi}} \text{Grad}_{\xi} K(\xi \cdot \eta) f(\eta) d\Gamma(\eta) + \\ &+ \int_{\partial C_{\xi}} K(\xi \cdot \eta) \sigma(\xi, \eta) f(\eta) d\partial C_{\xi}(\eta) \end{aligned} \quad (B.62)$$

where  $\sigma(\xi, \eta)$  denotes a unit vector, tangential to  $\Gamma$  in the point  $\xi$  and being in the plane spanned by  $\xi$  and  $\eta$ . Formula (B.62) is the result of the differentiation of a parameter integral where the parameter  $\xi$  occurs also in the domain.

Let  $\tilde{\lambda}_n$  denote the eigen values of the truncated kernel. Cf. (B.8-9). Then (B.61) may be written

$$\tilde{g}(\xi) = \sum_{n,m} \tilde{g}_{nm} S_{nm}(\xi) = \sum_{n,m} \tilde{\lambda}_n f_{nm} S_{nm}(\xi) \quad (B.64)$$

From (B.60) follows then

$$\tilde{w}(\xi) = \sum_{n,m} \tilde{g}_{nm} \text{Grad } S_{nm}(\xi) = \sum_{n,m} \tilde{\lambda}_n f_{nm} \text{Grad } S_{nm}(\xi) \quad (B.65)$$

This means that  $\tilde{g}_{nm} = \tilde{\lambda}_n f_{nm}$  are the coefficients of  $\tilde{w}(\xi)$  with respect to the system  $\text{Grad } S_{nm}(\xi)$ . This system is orthogonal. Cf. Meissl (1971), section 2. It is, however, not normalized, even if the  $S_{nm}$  are. If the  $S_{nm}$  are normalized in the usual way, then the mean square norm of  $\tilde{g}(\xi)$  is given by

$$\|\tilde{g}\|^2 = \int_{\Gamma} \tilde{g}^2(\xi) d\Gamma(\xi) = 4\pi \sum_{n,m} \tilde{g}_{nm}^2 = 4\pi \sum_{n,m} (\tilde{\lambda}_n f_{nm})^2 \quad (B.66)$$

and that of  $\tilde{w}(\xi)$  is given by

$$\|\tilde{w}\|^2 = \int_{\Gamma} (\tilde{w}(\xi), \tilde{w}(\xi)) d\Gamma(\xi) = 4\pi \sum_{n,m} n(n+1) (\tilde{\lambda}_n f_{nm})^2 \quad (B.67)$$

This is more fully explained in the stated reference.

Because of the linearity structure the above discussion carries immediately over to the truncation errors.

$$\Delta g(\xi) = g(\xi) - \tilde{g}(\xi) \quad , \quad \Delta w(\xi) = w(\xi) - \tilde{w}(\xi)$$

Let  $\Delta\lambda_n$  denote the eigen-values of the error kernel as in (B.13). Then

$$\Delta g(\xi) = \sum_{n,m} \Delta g_{nm} S_{nm}(\xi) = \sum_{n,m} \Delta \lambda_n f_{nm} S_{nm}(\xi) \quad (B.64a)$$

$$\Delta w(\xi) = \sum_{n,m} \Delta g_{nm} \text{Grad } S_{nm}(\xi) = \sum_{n,m} \Delta \lambda_n f_{nm} \text{Grad } S_{nm}(\xi) \quad (B.65a)$$

$$\|\Delta g\|^2 = \int_{\Gamma} \Delta g^2(\xi) d\Gamma(\xi) = 4\pi \sum_{n,m} \Delta g_{nm}^2 = 4\pi \sum_{n,m} (\Delta \lambda_n f_{nm})^2 \quad (B.66a)$$

$$\|\Delta w\|^2 = \int_{\Gamma} (\Delta w(\xi), \Delta w(\xi)) d\Gamma(\xi) = 4\pi \sum_{n,m} n(n+1) (\Delta \lambda_n f_{nm})^2 \quad (B.67a)$$

The relationship (B.65) is quite simple and offers an easy discussion of the truncation errors. The usual truncation of the Vening-Meinesz integral is, however, not based on (B.62), but rather on

$$\widehat{w}(\xi) = \int_{C_{\xi}} \text{Grad}_{\xi} K(\xi, \eta) f(\eta) d\Gamma(\eta) \quad (B.68)$$

De Witte (1966) has noted this difference. He used associated Legendre-functions to deal with formula (B.68). In our treatment here a quite simple analytical expression for the difference between (B.68) and (B.62) is at hand:

$$\widehat{w}(\xi) = \widetilde{w}(\xi) - \int_{\partial C_{\xi}} K(\xi, \eta) \sigma(\xi, \eta) f(\eta) d\partial C_{\xi}(\eta) \quad (B.69)$$

This shows for the first that the difference vanishes if  $K(\cos \psi)$  happens to have zero at  $\psi = \psi_0$ . This is quite apparent from the graphs in De Witte (1966), though he was seemingly unaware of the simple relationship between the two different truncation procedures.

We can, of course, enforce a zero of the kernel at  $\psi_0$  by simply subtracting the constant  $k_0 = K(\cos \psi_0)$  as in the discussion around (B.51). The effect of this is a constant in (B.59) and zero in (B.60). A beneficial effect is noted if the formulas are truncated. As we have shown, the  $\Delta \lambda_n$  i.e. the eigen values of the error kernel

taper off more quickly. Moreover, the two quantities  $\hat{w}(\xi)$  and  $\tilde{w}(\xi)$  coincide. This shows that the truncation  $\tilde{w}$  can be considered as favorable too, at least for the higher harmonics. Equation (B.67a) shows that the higher harmonic contribution to  $\Delta\hat{w} = \Delta w$  is in the same dampened by the factor  $\Delta\lambda$  as in the case of  $\Delta\tilde{g}$ .

If we deal with Stokes and Vening-Meinesz formulas then some constant factors enter the discussion. For we have

$$\text{Stokes: } N(\xi) = \frac{R}{4\pi G} \int_{\Gamma} \text{St}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (\text{B.70})$$

Vening-Meinesz:

$$v(\xi) = - \frac{1}{4\pi G} \int_{\Gamma} \text{Grad}_{\xi} \text{St}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (\text{B.71})$$

These factors are easily taken care of.

Summarizing we may say the following: Adding the constant  $-\text{St}(\cos \psi_0)$  to Stokes' kernel causes a constant added to the undulation  $N(\xi)$  which is of no concern since the geoid has to be scaled later on anyway. The added constant drops out completely in the Vening-Meinesz formula. If the formulas are truncated at  $\psi_0$ , then the higher harmonics of the truncation error are favorably dampened. Moreover the usual truncation of the Vening-Meinesz formula yields exactly the slope of the geoid computed from the truncated kernel  $\text{St}(\xi \cdot \eta) - \text{St}(\cos \varphi_0)$ .

If the Stokes' kernel is truncated over an interval  $\psi_1 \leq \psi \leq \psi_0$  so that the derivative at  $\psi = \psi_0$  also vanishes, then the same can be said. The dampening effect may be a little more favorable. The error discussion of the Vening-Meinesz formula can be based on the various quantities  $\Delta\lambda_n$  as they are discussed in sections B.4, B.5.

**B.6 Smoothing the Integrand in the Outer Zones.** Up to now we have been treating the case where the integrand is completely neglected outside a certain area. In this section we discuss a refinement in the sense that the integrand is not replaced by zero but rather by a smoothed version like a moving average over a certain area.

Let us start with formula (B.1) which we write now as

$$g^{(0)}(\xi) = \int_{\Gamma} K^{(0)}(\xi \cdot \eta) f^{(0)}(\eta) d\Gamma(\eta) \quad (\text{B.72})$$

Assume truncation (in some way) at an angle  $\psi_0$ . Denote by  $\tilde{K}^{(0)}(\xi \cdot \eta)$  the truncated kernel and by  $\Delta K^{(0)}(\xi \cdot \eta)$  the error or residual kernel. Then:

$$K^{(0)}(\xi \cdot \eta) = \tilde{K}^{(0)}(\xi \cdot \eta) + \Delta K^{(0)}(\xi \cdot \eta) \quad (\text{B.73})$$

The error is then

$$\Delta g^{(0)}(\xi) = \int_{\Gamma} \Delta K^{(0)}(\xi \cdot \eta) f^{(0)}(\eta) d\Gamma(\eta) \quad (\text{B.74})$$

or according to section B.1:

$$\Delta g_{nm}^{(0)} = \Delta \lambda_n^{(0)} f_{nm}^{(0)} \quad (\text{B.75})$$

with

$$\Delta \lambda_n^{(0)} = \int_{\Gamma} \Delta K^{(0)}(\xi \cdot \eta) P_n(\xi \cdot \eta) d\Gamma(\eta) \quad (\text{B.76})$$

The shape of the error kernel depends on the philosophy of the truncation procedure. Confer sections B.3 to B.4.

Let us account for the truncation error by computing the correction term (B.74). If we do this by using  $f^{(0)}(\xi)$  in its original form then of course no truncation error remains. We could have used the non-truncated kernel in the first place. The point is, however, that in the correction formula (B.74) a simplified version of  $f^{(0)}(\xi)$  may be used. Let us use  $f^{(1)}(\xi)$  instead of  $f^{(0)}(\xi)$  in (B.74) and call

$$\Delta f^{(0)}(\xi) = f^{(0)}(\xi) - f^{(1)}(\xi) \quad (\text{B.77})$$

then a residual error of

$$\Delta g^{(1)}(\xi) = \int_{\Gamma} \Delta K^{(0)}(\xi \cdot \eta) \Delta f^{(0)}(\eta) d\Gamma(\eta) \quad (\text{B.78})$$

remains. The spherical harmonics of this error can be computed by

$$\Delta g_{nm}^{(1)} = \Delta \lambda_n^{(0)} \Delta f_{nm}^{(0)} \quad (B.79)$$

Assume that  $f^{(1)}(\xi)$  is the result of the application of an isotropic smoothing operator toward  $f^{(0)}(\xi)$ . If  $\beta_n^{(0)}$  are the eigen values of this smoothing operator, then we have

$$f_{nm}^{(1)} = \beta_n^{(0)} f_{nm}^{(0)} \quad (B.80)$$

and consequently

$$\Delta f_{nm}^{(0)} = (1 - \beta_n^{(0)}) f_{nm}^{(0)} = \Delta \beta_n^{(0)} f_{nm}^{(0)} \quad (B.81)$$

so that (B.79) becomes

$$\Delta g_{nm}^{(1)} = \Delta \lambda_n^{(0)} \Delta \beta_n^{(0)} f_{nm}^{(0)} \quad (B.82)$$

The  $\Delta \lambda_n^{(0)}$  generally taper off with increasing  $n$  (though they may oscillate while doing this). From a certain  $n$  on they will be negligible under any circumstances. It is therefore desirable to have small  $\Delta \beta_n^{(0)}$  for moderately large  $n$ . This means one should have  $\beta_n^{(0)}$  close to 1 for moderately large  $n$ . One way is to put  $\beta_n^{(0)} = 1$  for certain  $n \leq n_0$  and  $\beta_n^{(0)} = 0$  for  $n > n_0$ . This amounts to a replacement of  $f^{(0)}(\xi)$  by its spherical harmonic expansion  $f^{(1)}(\xi)$  truncated at  $n = n_0$ .

More common is another smoothed version; namely, averages over certain block areas. A theoretical discussion of this is somewhat complicated since this smoothing operator is; 1) not isotropic and 2) applied only at discrete locations (which may be taken at the mid points of the blocks). Easier is a discussion based on moving averages over circular caps of half opening angle  $\alpha_0$ . For error consideration it is justifiable to replace the block averages by moving averages over circular caps of comparable size.

The eigen values  $\beta_n^{(0)}$  of these operators are found in Meissl (1971), equ. (3.14a), (3.14b). See also Pellinen (1966). We use the above cited formula (3.14a):

$$\beta_n^{(0)} = \frac{1}{1 - \cos \alpha_0} \int_{\cos \alpha_0}^1 P_n(t) dt \quad (\text{B.83})$$

For moderate large  $\alpha_0$  and  $n$  we may replace  $P_n(t)$  by its Taylor-linearization at  $t = 1$ :

$$P_n(t) = 1 - \frac{n(n+1)}{2} (1-t) + O((1-t)^2) \quad (\text{B.84})$$

This leads in a straight-forward way to

$$\beta_n^{(0)} = 1 - \frac{n(n+1)}{4} (1 - \cos \alpha_0) + O((1 - \cos \alpha_0)^2) \quad (\text{B.85})$$

Neglecting the 0-term we have

$$\beta_n^{(0)} \approx 1 - \frac{n(n+1)}{4} (1 - \cos \alpha_0) \quad (\text{B.86})$$

For the purpose of error estimates this formula may be used as long as

$$\beta_n^{(0)} \geq 0.80, \text{ say.}$$

Formula (B.72) is now replaced by

$$\begin{aligned} g^{(0)}(\xi) &= \int_{\Gamma} K^{(0)}(\xi \cdot \eta) f^{(0)}(\eta) d\Gamma(\eta) = \\ &= \int_{\Gamma} \tilde{K}^{(0)}(\xi \cdot \eta) f^{(0)}(\eta) d\Gamma(\eta) + \\ &+ \int_{\Gamma} \Delta K^{(0)}(\xi \cdot \eta) f^{(1)}(\eta) d\Gamma(\eta) + \Delta g^{(1)}(\xi) \end{aligned} \quad (\text{B.87})$$

with

$$\Delta g^{(1)}(\xi) = \int_{\Gamma} \Delta K^{(0)}(\xi \cdot \eta) \Delta f^{(0)}(\eta) d\Gamma(\eta) \quad (\text{B.88})$$

The following notations have been adopted

$K^{(0)}(\xi \cdot \eta)$	....	original kernel
$\tilde{K}^{(0)}(\xi \cdot \eta)$	....	truncated kernel
$\Delta K^{(0)}(\xi \cdot \eta)$	....	residual kernel: $K^{(0)}(\xi \cdot \eta) - \tilde{K}^{(0)}(\xi \cdot \eta)$
$f^{(0)}(\xi)$	....	original function
$f^{(1)}(\xi)$	....	smoothed version of function
$\Delta f^{(0)}(\xi)$	....	residual $f^{(0)}(\xi) - f^{(1)}(\xi)$

It is considered that the first two terms on the right-hand side in (B.87) are computed. The third term  $\Delta g^{(1)}(\xi)$  represents the error term (B.88) which in spherical harmonics representation is given by (B.82).

One can, however, go a step further and try to truncate  $\Delta K^{(0)}(\xi \cdot \eta)$  again further outside

$$\Delta K^{(0)}(\xi \cdot \eta) = \Delta \tilde{K}^{(0)}(\xi \cdot \eta) + \Delta K^{(1)}(\xi \cdot \eta) \quad (\text{B.89})$$

The contribution of  $\Delta K^{(1)}(\xi \cdot \eta)$  could be taken into account but only after replacing  $f^{(1)}(\xi)$  by a still heavier smoothed version  $f^{(2)}(\xi)$ .

There is no need for a detailed discussion of the further procedure which could go on and on, replacing  $f^{(0)}(\xi)$  by smoother and smoother versions in a succession of concentric zones. We shall restrict us here the following remarks:

(1): If truncation over an interval is used, then there is an overlapping of neighboring zones. This does not matter if it is properly taken care of.

(2): One should avoid adding constants toward the  $\Delta K^{(j)}$  kernels before truncation as it has in some cases been done for the original kernel  $K^0(\xi \cdot \eta)$ . This would make the kernels non-zero also in zones farther inside. If  $\Delta K^{(j)}$  happens to have a zero at a location which is suitable for truncation then truncation is advantageous in the same way as it was for a zero of  $K(\cos \psi)$ . Kernels from which the harmonic components up to degree  $N$  have been removed have at least  $N + 1$  zeros. (Meissl (1971)). These zeros are natural locations for truncation, or for a transition to a heavier smoothed version  $f^{(j+1)}(\xi)$  of  $f^{(j)}(\xi)$ , respectively.

## APPENDIX C: Regularization of a Type M-Integral

What we call a type M-integral looks like

$$g(\xi) = \int_{\Gamma} \frac{\varphi(\xi, \eta)}{\ell^3(\xi \cdot \eta)} d\Gamma(\eta) \quad (C.1)$$

It is singular and we require that

$$\varphi(\xi, \xi) = 0 \quad (C.2)$$

and also that  $\varphi(\xi, \eta)$  is twice continuously differentiable throughout.

Examples are

$$g(\xi) = \int_{\Gamma} \frac{f(\eta) - f(\xi)}{\ell^3(\xi \cdot \eta)} d\Gamma(\eta) \quad (C.3)$$

or

$$g(\xi) = \int_{\Gamma} \frac{f(\eta) - f(\xi)}{\ell^3(\xi \cdot \eta)} h(\eta) d\Gamma(\eta) \quad (C.4)$$

The latter is the correction term in the first order solution of Molodensky's problem. Therefore the name "type-M" has been chosen.

The regularization procedure will be based on Green's second formula. Only a cap  $C$  will be considered since for larger distances the integral tapers off quickly. There the original form may be used. Complete truncation at a certain distance is also feasible in most applications.

The contribution of a spherical cap  $C$  centered at  $\xi$  and having half-opening angle  $\psi_0$  to a type-M integral is

$$g_C(\xi) = \int_C \frac{\varphi(\xi, \eta)}{\ell^3(\xi \cdot \eta)} d\Gamma(\eta) \quad (C.5)$$

Formula (B.21) allows us to rewrite this as

$$\begin{aligned}
g_C(\xi) = g_1(\xi) + g_2(\xi) &= \int_C \text{Lap}_\eta \frac{1}{l(\xi \cdot \eta)} \varphi(\xi, \eta) d\Gamma(\eta) \\
&- \int_C \frac{1}{4} \frac{1}{l(\xi \cdot \eta)} \varphi(\xi, \eta) d\Gamma(\eta)
\end{aligned} \tag{C.6}$$

The second integral, i.e. that one for  $g_2(\xi)$  is already regular. In some applications it may even be omitted since after multiplication by  $R^{-1}$  it is negligible in the so-called planar approximation. See Moritz (1969).

Before we apply Green's second formula toward  $g_2(\xi)$  we exclude a small cap  $C_1$  around  $\xi$  of half opening angle  $\psi_1$ . We have then

$$\begin{aligned}
\tilde{g}_1(\xi) &= \int_{C-C_1} \text{Lap}_\eta \frac{1}{l(\xi \cdot \eta)} \varphi(\xi, \eta) d\Gamma(\eta) = \\
&= \int_{C-C_1} \frac{1}{l(\xi \cdot \eta)} \text{Lap}_\eta \varphi(\xi, \eta) d\Gamma(\eta) + \\
&+ \oint_{\partial C} \varphi(\xi, \eta) \left( \text{Grad}_\eta \frac{1}{l(\xi \cdot \eta)}, \nu(\eta) \right) d\partial C(\eta) - \\
&- \oint_{\partial C} \frac{1}{l(\xi \cdot \eta)} \left( \text{Grad}_\eta \varphi(\xi, \eta), \nu(\eta) \right) d\partial C(\eta) - \\
&- \oint_{\partial C_1} \varphi(\xi, \eta) \left( \text{Grad}_\eta \frac{1}{l(\xi \cdot \eta)}, \nu_1(\eta) \right) d\partial C_1(\eta) + \\
&+ \oint_{\partial C_1} \frac{1}{l(\xi \cdot \eta)} \left( \text{Grad}_\eta \varphi(\xi, \eta), \nu_1(\eta) \right) d\partial C_1(\eta)
\end{aligned} \tag{C.7}$$

$\nu(\eta)$  is (outward) normal to the boundary of  $C$  and so is  $\nu_1(\eta)$  to the boundary of  $C_1$ . If we let  $C_1$  contract toward the point  $\xi$  then it may be shown that the last two integrals vanish because of the assumptions regarding  $\varphi(\xi, \eta)$ . The third integral on the right-hand side may be combined with the first one. This is possible since

$l(\xi \cdot \eta)$  is constant and equal to  $l(\cos \psi_0)$  in the third integral, so that Gauss' integral theorem may be applied. We get then

$$g_1(\xi) = \int_C \left\{ \frac{1}{l(\xi \cdot \eta)} - \frac{1}{l(\cos \psi_0)} \right\} \text{Lap}_\eta \varphi(\xi, \eta) d\Gamma(\eta) \\ - \frac{\cos \frac{\psi_0}{2}}{l^2(\cos \psi_0)} \oint_{\partial C} \varphi(\xi, \eta) d\partial \cap \eta \quad (C.8)$$

Use has also been made of

$$(\text{Grad}_\eta \frac{1}{l(\xi \cdot \eta)}, \nu(\eta)) = - \frac{\cos \frac{\psi_0}{2}}{l^2(\xi \cdot \eta)} \quad (C.9)$$

In a local  $\psi, \alpha$  system, the formula for  $g_1$  would read

$$g_1 = \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} \left\{ \frac{1}{l(\cos \psi)} - \frac{1}{l(\cos \psi_0)} \right\} \text{Lap} \varphi(\psi, \alpha) \sin \psi d\psi d\alpha \\ - \left\{ \frac{1}{l(\cos \psi_0)} - \frac{l(\cos \psi_0)}{4} \right\} \int_{\alpha=0}^{2\pi} \varphi(\psi_0, \alpha) d\alpha \quad (C.10)$$

This completes the regularization of the M-type integral for a spherical cap.

#### APPENDIX D: Regularization of the Vening-Meinesz Formula

We transform the Vening-Meinesz formula in a way that the resulting integrals are no longer singular. We deal thereby only with the contribution of a circular cap  $C$  around the point  $\xi$  in which the deflection is to be computed. The half-opening angle of this cap shall be  $\psi_0$ . For the more remote zones the original form of Vening-Meinesz formula is of greater advantage since the kernel tapers off quickly with increasing distance.

The contribution of this cap to the Vening-Meinesz formula is

$$v_C(\xi) = - \frac{1}{4\pi G} \int_C \text{Grad}_\xi \text{St}(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (\text{D.1})$$

The singularity in the Vening-Meinesz formula stems from the term  $1/\sin \frac{\psi}{2}$  in Stokes' kernel (B.24). This term also equals  $2/l(\xi \cdot \eta)$ . If we split therefore

$$\text{St}(\xi \cdot \eta) = \frac{2}{l(\xi \cdot \eta)} + R(\xi \cdot \eta) \quad (\text{D.2})$$

then the contribution of  $R(\xi \cdot \eta)$  leads to a regular integral

$$v_2(\xi) = - \frac{1}{4\pi G} \int_C \text{Grad}_\xi R(\xi \cdot \eta) \Delta g(\eta) d\Gamma(\eta) \quad (\text{D.3})$$

The contribution of  $2/l(\xi \cdot \eta)$  has to be dealt with further:

$$v_1(\xi) = - \frac{1}{2\pi G} \int_C \text{Grad}_\xi \frac{1}{l(\xi \cdot \eta)} \Delta g(\eta) d\Gamma(\eta) \quad (\text{D.4})$$

If this is done in a satisfactory way, then

$$v_C(\xi) = v_1(\xi) + v_2(\xi) \quad (\text{D.5})$$

We need now some vector analytical preparations. Call  $\sigma(\xi, \eta)$  a unit vector which is tangential to  $\Gamma$  in the point  $\xi$  and is contained in the plane spanned by  $\xi$

and  $\eta$ . In other words,  $\sigma(\xi, \eta)$  is orthogonal to  $\xi$  and coplanar with  $\xi$  and  $\eta$ . Call  $\tau(\xi, \eta)$  the vector  $\xi \times \sigma(\xi, \eta)$ . We view  $\sigma(\xi, \eta)$  as a vector which is imbedded in 3-space. Then the operators  $\text{Grad}_\eta$  and  $\text{Lap}_\eta$  can be applied to each of the three components. The following can be verified ( $\cos \psi = \xi \cdot \eta$ )

$$\text{Grad}_\eta \sigma(\xi, \eta) = \frac{1}{\sin \psi} \tau(\xi, \eta) \tau(\xi, \eta)^T \quad (\text{D.6})$$

$$\text{Lap}_\eta \sigma(\xi, \eta) = - \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) \quad (\text{D.7})$$

We do not give a detailed derivation of these formulas. Chapter 6 in Meissl (1971a) deals more extensively with the underlying concepts.

Performing the differentiation in (D.4) gives

$$v_1(\xi) = - \frac{1}{2\pi G} \int_C \frac{\cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) \Delta g(\eta) d\Gamma(\eta)$$

Hence by (D.7)

$$\begin{aligned} v_1(\xi) &= \frac{1}{2\pi G} \int_C \text{Lap}_\eta \sigma(\xi, \eta) \Delta g(\eta) d\Gamma(\eta) + \\ &\quad + \frac{1}{2\pi G} \int_C \frac{1 - \cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) \Delta g(\eta) d\Gamma(\eta) \\ &= v_{11}(\xi) + v_{12}(\xi), \text{ say.} \end{aligned} \quad (\text{D.8})$$

$v_{12}(\xi)$  is already regular. The integrand is even bounded. We apply Green's second formula (component-wise) toward  $v_{11}(\xi)$  and obtain:

$$\begin{aligned}
v_{11}(\xi) &= \frac{1}{2\pi G} \int_C \sigma(\xi, \eta) \text{Lap } \Delta g(\eta) d\Gamma(\eta) - \\
&- \frac{1}{2\pi G} \oint_{\partial C} \sigma(\xi, \eta) (\text{Grad } \Delta g(\eta), \nu(\eta)) d\partial C(\eta) + \\
&+ \frac{1}{2\pi G} \oint_{\partial C} \Delta g(\eta) (\text{Grad}_\eta \sigma(\xi, \eta), \nu(\eta)) d\partial C(\eta)
\end{aligned}$$

The last integral vanishes because of (D.6) and the orthogonality of  $\tau(\xi, \eta)$  and  $\nu(\eta)$ . Summarizing we have the following:

$$\begin{aligned}
v_C(\xi) &= \frac{1}{2\pi G} \int_C \sigma(\xi, \eta) \text{Lap } \Delta g(\eta) d\Gamma(\eta) - \\
&- \frac{1}{2\pi G} \oint_{\partial C} \sigma(\xi, \eta) (\text{Grad } \Delta g(\eta), \nu(\eta)) d\partial C(\eta) + \\
&+ \frac{1}{2\pi G} \int_C \frac{1 - \cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) \Delta g(\eta) d\Gamma(\eta) - \\
&- \frac{1}{4\pi G} \int_C \text{Grad}_\xi R(\xi, \eta) \Delta g(\eta) d\Gamma(\eta)
\end{aligned} \tag{D.9}$$

In the literature, the Vening-Meinesz formula is usually written in a local  $\psi, \alpha$  coordinate system. In this notation (D.9) reads as:

$$\begin{aligned}
v_C &= \frac{1}{2\pi G} \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \text{Lap } \Delta g(\psi, \alpha) \sin \psi d\psi d\alpha - \\
&- \frac{1}{2\pi G} \sin \psi_0 \int_{\alpha=0}^{2\pi} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \left. \frac{\partial \Delta g(\psi, \alpha)}{\partial \psi} \right|_{\psi=\psi_0} d\alpha +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\pi G} \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} \frac{1 - \cos^3 \frac{\psi}{2}}{\sin \psi} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \Delta g(\psi, \alpha) d\psi d\alpha - \\
& + \frac{1}{4\pi G} \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} \frac{\partial R(\cos \psi)}{\partial \psi} \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \Delta g(\psi, \alpha) \sin \psi d\psi d\alpha \\
& \dots (D.9a)
\end{aligned}$$

In the main portion of this report the Vening-Meinesz formula occurs in (1.2).  $\Delta g(\eta)$  has then to be replaced by  $\Delta g(\eta)$  plus some correction terms.

APPENDIX E : Regularizing a Correction Term in the Deflection - Formula.

We shall be concerned with the following term contributing toward the deflection formula (1.2) :

$$q(\xi) = \frac{1}{4\pi R^2 G} \int_{\Gamma} (h(\eta) - h(\xi))^2 \text{Grad}_{\xi} \frac{1}{\ell^3(\xi, \eta)} \Delta g(\eta) d\Gamma(\eta) \quad \dots (E.1)$$

Using

$$\text{Grad}_{\xi} \frac{1}{\ell^3(\xi, \eta)} = \frac{3 \cos^5 \frac{\psi}{2}}{\sin^4 \psi} \sigma(\xi, \eta) \quad (E.2)$$

and

$$\text{Lap}_{\eta} \left\{ \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) \right\} = \left\{ \frac{3}{\sin^4 \psi} - \frac{2}{\sin^2 \psi} \right\} \sigma(\xi, \eta) \quad (E.3)$$

we may write the cap contribution of (E.1) as :

$$\begin{aligned} q_C(\xi) &= \frac{1}{4\pi R^2 G} \int_C [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \text{Lap}_{\eta} \left\{ \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) \right\} d\Gamma(\eta) \\ &\quad - \frac{1}{4\pi R^2 G} \int_C [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \left\{ \frac{3(1 - \cos^5 \frac{\psi}{2})}{\sin^4 \psi} - \right. \\ &\quad \left. - \frac{2}{\sin^2 \psi} \right\} \sigma(\xi, \eta) d\Gamma(\eta) \\ &= q_1(\xi) + q_2(\xi) , \quad \text{say.} \end{aligned} \quad (E.4)$$

$q_2(\xi)$  is already regular. Toward  $q_1(\xi)$  Greens second formula will be applied yielding

$$\begin{aligned}
q_1(\xi) = & \frac{1}{4\pi R^2 G} \int_C \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) \text{Lap}_\eta [(h(\eta) - h(\xi))^2 \Delta g(\eta)] d\Gamma(\eta) \\
& - \frac{1}{4\pi R^2 G} \oint_{\partial C} \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) ( \text{Grad}_\eta [(h(\eta) - h(\xi))^2 \Delta g(\eta), \\
& \quad , v(\eta)) d\partial C(\eta) \\
& - \frac{1}{2\pi R^2 G} \oint_{\partial C} [(h(\eta) - h(\xi))^2 \Delta g(\eta)] \frac{\cos \psi}{\sin^3 \psi} \sigma(\xi, \eta) d\partial C(\eta) \\
& \dots (E.5)
\end{aligned}$$

For the last term (D.6) has been used. The applicability of Greens second formula has to be justified in the usual way (letting another small cap contract toward  $\xi$ ).

The first term in (E.5) contains still a singular integral. We have:

$$\begin{aligned}
\text{Lap}_\eta [(h(\eta) - h(\xi))^2 \Delta g(\eta)] = & 2 |\text{Grad } h(\eta)|^2 \Delta g(\eta) \text{ plus} \\
& \text{terms containing } (h(\eta) - h(\xi)) \text{ as a factor.} \quad (E.6)
\end{aligned}$$

The terms containing  $(h(\eta) - h(\xi))$  as a factor cause no singularity problem. However the term  $2|\text{Grad } h(\eta)|^2$  does. There is, however, no need for further regularization. The term

$$\frac{1}{2\pi R^2 G} \int_C \frac{1}{\sin^2 \psi} \sigma(\xi, \eta) |\text{Grad } h(\eta)|^2 \Delta g(\eta) d\Gamma(\eta) \quad (E.7)$$

can be combined with

$$- \frac{1}{2\pi R^2 G} \int_C \text{Grad} \frac{1}{\ell(\xi, \eta)} |\text{Grad } h(\eta)|^2 \Delta g(\eta) d\Gamma(\eta) \quad (E.8)$$

in a way that the singularity cancels. (E.8) is contributing

to the first term of the deflection formula (1.2). To see this consider the cap contribution of (1.2), split  $St(\xi)$  according to (D.2) and note the definition of  $G_2(\xi)$  in (1.4). The result of combining (E.7) and (E.8) is

$$\frac{1}{2\pi R^2 G} \int_C \frac{1 - \cos^3 \frac{\psi}{2}}{\sin^2 \psi} \sigma(\xi, \eta) |\text{Grad } h(\eta)|^2 \Delta g(\eta) d\Gamma(\eta) \dots (E.9)$$

which is now regular.

The result of this somewhat involved regularization procedure is summarized in section 3.5.4.

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